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Bachelor Thesis

Modelling and optimisation of multi-stage water rockets

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Contents

I. Motivation

Whether its an antique steam locomotive in the basement or the newest Formula One racing car on the living room's racetrack, scaling complex technical systems down to manageable sizes and principles fascinate people of all generations. And what can be more inspiring than building a rocket that can fly right into sky? In the simplest approach one only needs an empty water bottle, an air-pump, a cork and some amount of water to get a little closer to the stars. For this reason water rockets are very often seen as toys, but this is misleading: a description of all physics elements involved gets rather complex.

In this thesis I will approach to this description on a computational way. In the first part I want to illustrate how a reasonable, but still manageable physical model of a water rocket can be formulated. After being able to simulate such a rocket the task of the second part will be its optimisation. Since all rocket enthusiasts strive for higher altitude the main goal will be to reach higher apogees with the same rocket by varying some of its parameters. In addition it will be analysed how small changes of single parameters affect a water rocket. By doing all this hopefully a deeper insight into the physics and the behaviour of a water rocket will be accomplished.

Figure 1.1.: Advanced Water Rocket Axion V [1]

II. Modelling

2.1. Overview

In this chapter the different modules of a water rocket and the resulting integrated model will be presented. At the end simplifications and shortcomings of the model and advantages of adding more stages will be discussed.
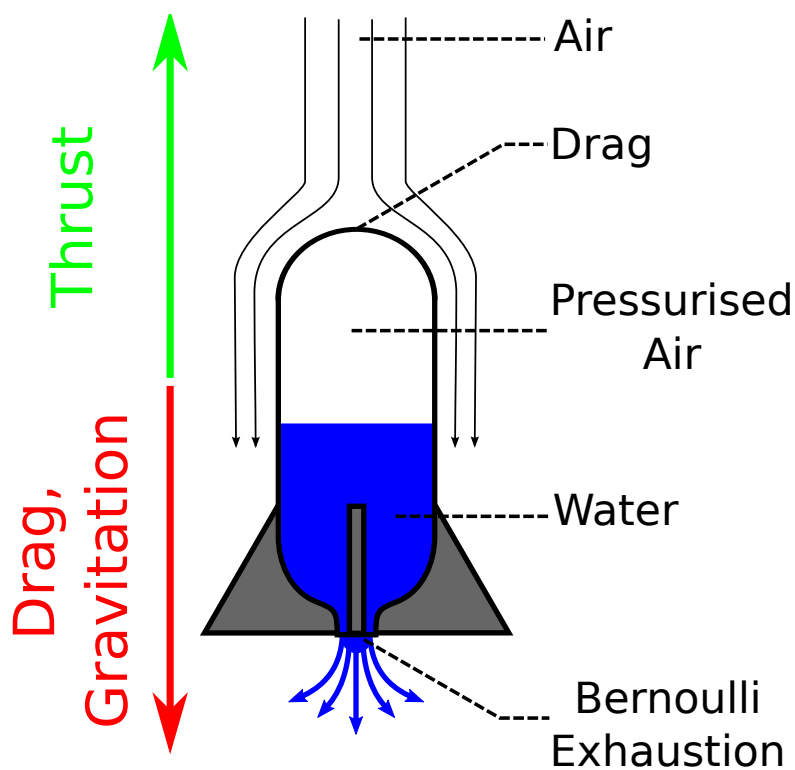

Figure 2.1.: Water rocket modules

Figure 2.1 shows the different forces $\vec{F}_i$ acting on a water rocket defining the modules of the integrated model. This model allows to obtain the numerical solution of the system and to characterise the performance of a rocket.

$$\vec{F}_R = m_R \cdot \frac{dv_R}{dt} = \vec{F}_{Th} + \vec{F}_G + \vec{F}_D \tag{2.1}$$

2.2. Modules

Here the individual modules are introduced and merged into a model.

2.2.1. Thrust

The thrust $\vec{F}_{Th}$ produced by a water rocket is given by the well known Tsiolkovsky rocket equation [2]

$$\vec{F}_{Th}(t) = \vec{v}_e(t) \cdot \frac{dm_R}{dt} \ , \tag{2.2}$$

in which the exhaust velocity at the nozzle $\vec{v}_e(t)$ and the time derivative of the total rocket mass $m_R(t)$ yet has to be determined. Here I assume that the acceleration process can be split into two parts. In the first one no air leaves the bottle while the water is pressed out. After that the remaining pressurised air expands very fast, giving the rocket another short amount of thrust. By this separation equation (2.2) splits into two equations, which are later used after each other in the simulation according to the acceleration phase.

The key issue for calculating the thrust is to understand how the air inside the rocket expands. Therefore, the needed functions are mainly derived according to Shaviv [3] and Finney [4]. Since the typical time scale of the expansion is only about some tenths of a second and PET has a rather small thermal conductivity the process is assumed to be adiabatic. Due to the fact that temperatures and pressures inside an ordinary water rocket are far away from the critical point of air, the gas can be treated as ideal. Based on these assumptions, one can use basic thermodynamic equations to obtain the needed functions for equation (2.2).

Water thrust

The process of water getting pressed out of an opening can be described by Bernoulli's equation [5]

$$p(t) + \frac{1}{2}\rho_W v_{in}^2 = p_0 + \rho_W \frac{1}{2} v_E^2 \ , \tag{2.3}$$

with the pressure inside the rocket $p(t)$, the atmospheric pressure p_0, the water density ρ_W and the velocity of the water inside the bottle v_{in}, relative to the whole rocket. Because the effective area at the nozzle is much smaller the water moves much faster there and the term describing the flow of water inside the rocket can be neglected. One achieves the following equation for the exhaust velocity:

$$v_E(t) = \sqrt{\frac{2 \cdot (p(t) - p_0)}{\rho_W}} \ . \tag{2.4}$$

Obviously, the next step will be to determine the pressure inside the bottle as a function of time. As known from thermodynamics and visualised in figure 2.2 the
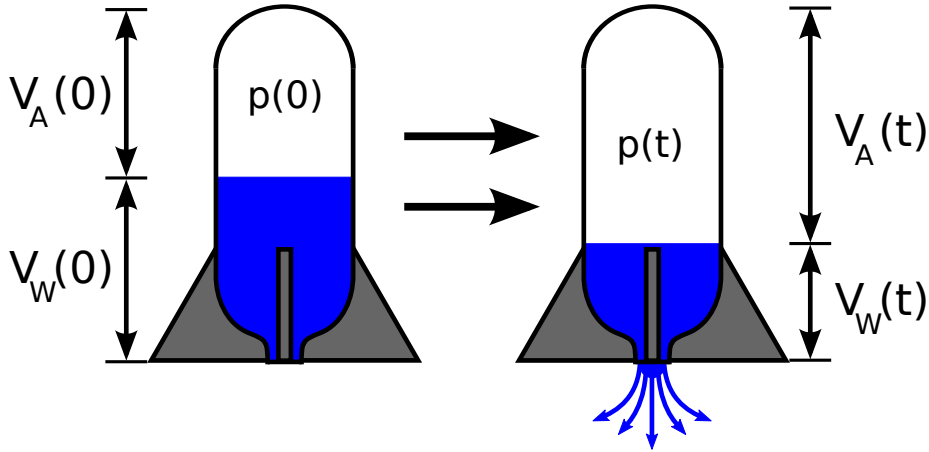
Figure 2.2.: Air expansion inside the rocket

gas expands from an initial pressure $p(0)$ and volume $V_A(0)$ to another state in time $p(t)$ and $V_A(t)$, fulfilling the following condition in which γ is the adiabatic coefficient:

$$p(0)V_A^\gamma(0) = p(t)V_A^\gamma(t)$$

$$\Rightarrow p(t) = p(0) \left(\frac{V_A(t)}{V_A(0)}\right)^{-\gamma} \quad . \tag{2.5}$$

Now, by splitting the air volume into $V_A(t) = V_A(0) + (V_W(0) - V_W(t))$ using the water volume $V_W(t)$, equation 2.5 becomes:

$$p(t) = p(0) \left(\frac{V_A(0) + (V_W(0) - V_W(t))}{V_A(0)}\right)^{-\gamma}$$

$$= p(0) \left(1 + \frac{m_W(0) - m_W(t)}{\rho_W V_A(0)}\right)^{-\gamma} \quad , \tag{2.6}$$

where $\rho_W = m_W(t)/V_W(t)$ is the water density. The total rocket mass $m_R(t) = m_W(t) + m_A(t) + m_0$ consists of the water m_W and air mass m_A as well as the weight of the empty rocket itself m_0. Assuming that no air leaves the bottle during this process resulting in a constant $m_A(t)$, one obtains the final pressure relation.

$$p(t) = p(0) \left(1 + \frac{m_R(0) - m_A(0) - m_0 - m_R(t) + m_A(t) + m_0}{\rho_W V_A(0)}\right)^{-\gamma}$$

$$= p(0) \left(1 + \frac{m_R(0) - m_R(t)}{\rho_W V_A(0)}\right)^{-\gamma} \tag{2.7}$$

So, finding a function for the rocket mass is the last problem to be solved. That can easily be done by looking at the water mass flow rate at the nozzle with aperture area A_N:

$$\frac{dm_R}{dt} = \frac{dm_W}{dt} = \rho_W \cdot \frac{dV_W}{dt} = \rho_W A_N v_E(t) \quad . \tag{2.8}$$

Air thrust

After all water is pushed out the air inside the bottle might still be pressurised, now being free to expand very fast and therefore producing an additional amount of thrust. In general, this air expansion is very similar to the one with water, so the equation for the exhaust velocity has the same form as before.

$$v_E(t) = \sqrt{\frac{2 \cdot (p(t) - p_0)}{\rho_A}} \tag{2.9}$$

As one can see, the only difference is the replacement of the water density by the air density ρ_A. The adiabatic process needs more effort, since the air is expanding to a volume whose one part $V_{A,o}$ lies outside the bottle. This can be expressed by using the total rocket volume V_R as follows. It must be noted that this is a new process starting at a certain time (chosen as $t = t_0$ in the following equation).

$$\begin{aligned}
p(t) &= p(t_0) \left(\frac{V_A(t)}{V_A(t_0)} \right)^{-\gamma} \\
&= p(t_0) \left(\frac{V_R + V_{A,o}(t)}{V_R} \right)^{-\gamma} \\
&= p(t_0) \left(1 + \frac{m_{A,o}(t)}{\rho_A V_R} \right)^{-\gamma} \\
&= p(t_0) \left(1 + \frac{m_{A,i}(t_0) - m_{A,i}(t)}{\rho_A V_R} \right)^{-\gamma}
\end{aligned} \tag{2.10}$$

The indices i and o indicate mass or volume inside, respectively outside, the rocket. At last, there is only the function of the air mass inside the rocket left to determine. For the needed initial value $m_{A,i}(t_0)$ one has to note that the density is a function of pressure, identified with $\rho'_A(p) = p/(R_S T)$, where R is the specific gas constant and T the temperature. However, to calculate $m_{A,i}(t)$ it is not clear which pressure to use, since it is not certain how much the air has already expanded at the nozzle. I have used the pressure at standard conditions.

$$m_{A,i}(t_0) = m_{A,i}(0) = V_A(0)\rho'_A(p) = V_A(0)\frac{p(0)}{R_S T} \tag{2.11}$$

$$\frac{dm_{A,i}}{dt} = \rho_A A_N v_E(t) \tag{2.12}$$

2.2.2. Gravitation

One important force affecting the rocket trajectory is the gravitational force:

$$\vec{F}_G = m_R(t) \cdot \vec{g} \ . \tag{2.13}$$

The rocket mass has already been determined in the previous section and the gravitational acceleration g is assumed to be constant. This assumption will be further discussed in section 2.3.2.

2.2.3. Drag

The next contribution to the total force is the drag force acting on the rocket [6]:

$$F_D = \frac{1}{2} c_D \rho_A A v_R^2 \ . \tag{2.14}$$

Or to put it in a vectorial form:

$$\vec{F_d} = -\frac{1}{2} c_D \rho_A A (\vec{v_R} \cdot \vec{v_R}) \frac{\vec{v_R}}{\|\vec{v_R}\|} \ .$$

This form assures that the drag force always points against the velocity vector of the rocket and therefore not only restrains its ascent but also its descent. The determination of the drag coefficient c_D and the reference area A is a non-trivial problem, but was measured for model rockets in wind tunnel experiments [7].

2.2.4. Resulting model

To recap the results all equations describing this model are listed here:

$$m_R \cdot \frac{d\vec{v_R}}{dt} = \vec{v}_e(t) \cdot \frac{dm_R}{dt} + m_R(t) \cdot \vec{g} - \frac{1}{2} c_D \rho_A A (\vec{v_R} \cdot \vec{v_R}) \frac{\vec{v_R}}{\|\vec{v_R}\|} \ . \tag{2.15}$$

Water thrust phase:

$$v_E(t) = \sqrt{\frac{2 \cdot (p(t) - p_0)}{\rho_W}} \tag{2.16}$$

$$p(t) = p(0) \left(1 + \frac{m_W(0) - m_W(t)}{\rho_W V_A(0)} \right)^{-\gamma} \tag{2.17}$$

$$\frac{dm_R}{dt} = \rho_W A_N v_E(t) \tag{2.18}$$

Air thrust phase:

$$v_E(t) = \sqrt{\frac{2 \cdot (p(t) - p_0)}{\rho_A}} \tag{2.19}$$

$$p(t) = p(t_0) \left(1 + \frac{m_{A,i}(t_0) - m_{A,i}(t)}{\rho_A V_R} \right)^{-\gamma} \tag{2.20}$$

$$\frac{dm_{A,i}}{dt} = \rho_A A_N v_E(t) \tag{2.21}$$

2.3. Limitations

After defining the integrated model its shortcomings and limitations will be considered.

2.3.1. Wind effects

The developed model only considers the frontal drag, described in section 2.2.3. Further effects like crosswind are very difficult to include in their full complexity. For a self-consistent description one would have to solve the Navier-Stokes equations, which would go beyond the scope of this thesis. Therefore, these effects are completely ignored in the model. The question is what influence could such crosswinds have on a real model rocket. Firstly, it would simply interrupt the straight ascent of the rocket. All equations are assuming, that the acting forces are parallel or anti parallel to each other. If the rocket just tilts a little bit in the air, and this is what probably will happen if crosswind occurs, the whole force construction becomes inaccurate.
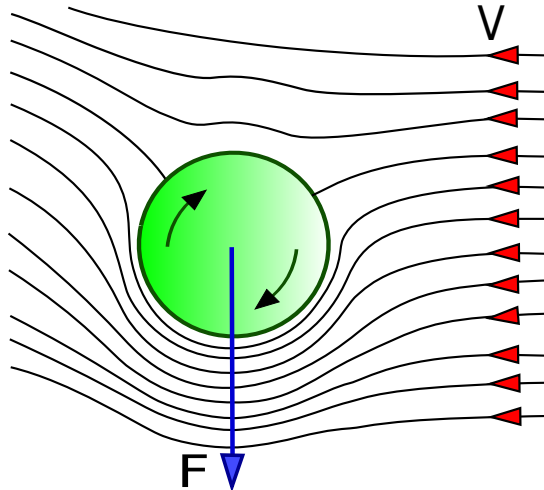

Figure 2.3.: Magnus effect [8]

Another force contribution neglected in this model is based on the Magnus effect. It describes a force which occurs on spinning objects moving through a fluid [9]. This force is always perpendicular to the direction of motion and the rotation axis. Frequently, model rockets are spinning while ascending, either as a consequence of the launch process or by purpose for stabilisation. So, if there is wind attacking the side of the rocket, the Magnus force will pull it sideways orthogonal to the wind direction (figure 2.3). As stated before this may tilt the rocket, which is beyond the scope of this model.

In summary, wind disturbs the rockets ascent very much in view of its uprightness. Therefore, I rank wind as a major source of errors in predicting the flight behaviour

of a model rocket. Unfortunately, it is also too difficult to consider it in detail in the model. Anyway, hobby astronauts would not start on windy days, risking the integrity of their model rocket, justifying this approximation.

2.3.2. Non-constant pressure and gravitation

As stated before, air pressure and gravitation decrease with height. Equations (2.16) and (2.19) show that a smaller air pressure at a given internal pressure results in a larger exhaust velocity and thrust. Also reduced gravitational acceleration apparently results in a smaller force as seen in equation (2.13). Both effects can cause the rocket to gain more altitude. To analyse the significance of both some estimates are used.

The pressure change is given by the barometric formula. Since normal water rockets only operate up to several hundred meters this exponential function can be linearised, which gives the approximation $\Delta p = 1/8 \text{ hPa/m} \cdot \Delta h$ [10]. A likewise simple approximation can be found for the gravitational acceleration: $\Delta g = 3.1 \mu\text{m/s}^2 \cdot \Delta h$ [11]. Assuming an internal pressure of 5000 hPa, a total rocket weight of 0.5 kg and a reached height of 200 m the differences of the model with and without these effects during the water thrust phase are rather small:

$$\Delta v_E = 620 \; \mu\text{m/s}$$
$$\Delta F_G = 310 \; \mu\text{N} \,.$$

Typical values for water rockets are $v_E = 20$ m/s, $F_G = 200$ N. Beside that, a falling air pressure also leads to a lower air density as already seen in equation (2.11). This would decrease the drag force in equation (2.14) about a small amount. With the previous parameters, an assumed drag coefficient of $c_D = 0.5$, a surface of $A = 0.01 \text{ m}^2$ and a rocket velocity of $v_R = 10$ m/s the difference would be only $\Delta F_D = 15$ mN, so that this effect is also negligible.

2.3.3. Water flow

Since the model is based on Bernoulli's equation its water outflow is assumed rather idealised as a laminar flow regime. The expanding air is seen as a piston, which pushes the water out of the bottle until no water is left inside. Afterwards, the remaining pressurized air exhausts. Unfortunately this is not supported by reality as shown in figure 2.4. These pictures have been taken at a water rocket workshop organised by Dr. Johannes Knapp at the University of Leeds. It looks like the outflow is first laminar as assumed in the model. But inside the rocket the water mixes with the air, resulting in a more turbulent outflow towards the end. So, a strict separation between water and air thrust phase seems rather unrealistic.

Also, in contrary to the piston analogy, the water has not a constant flow velocity inside the bottle. A realistic flow field is very complex and cannot be described by Bernoulli's equation. It leads to more drag inside the bottle and therefore to lower
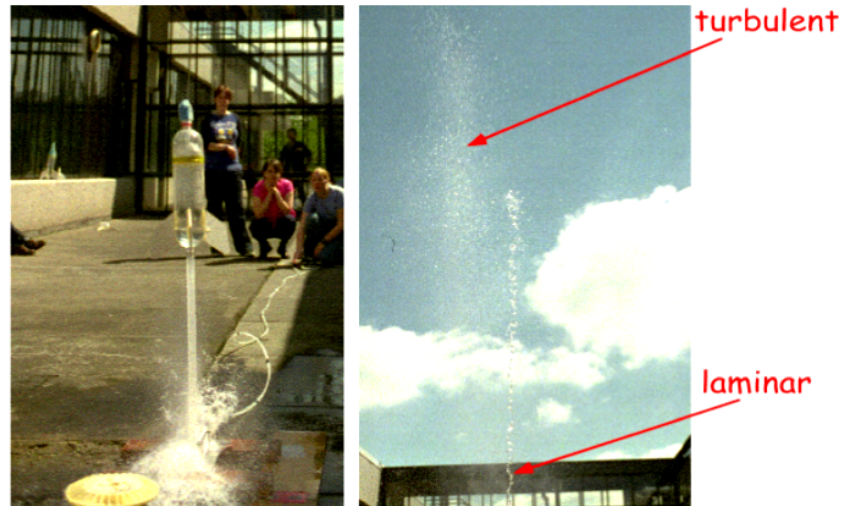
Figure 2.4.: Water exhaust [12]

outflow velocities. Therefore, in addition to the wind the water flow is another major uncertainty factor in the model, which has to be ignored, since simulation of turbulent water is a very difficult task involving a more complex model.

2.3.4. Imbalances

As already stated, the model assumes parallel or anti-parallel forces. Beside the wind, imbalances of the rocket itself may also invalidate this assumption. As the water flows out of the bottle the centre of gravity is moving upwards. The instability of the rocket and its urge to tilt rises as the tank gets empty. Construction-inaccuracies may also dislocate the centre of gravity sideways forcing the rocket to tilt by itself. In relation to these problems the model is rather an optimistic prediction, which can be reached by clever design.

2.4. Multi-staging

After deriving the equations for the model, it is obvious that a single stage rocket has very strict limitations. For instance, a higher pressure inside the bottle is always resulting in more thrust practically without any negative effect. Of course, at some point, the weight gain due to the pressurised air may become significant, but much earlier the bottle will simply burst. Also, a higher volume seems superior, but in consequence this might decrease the structural integrity of the bottle and therefore limit the maximum pressure.

As described by Turner [13] or Messerschmid and Fasoulas [2] it is a very intuitive step to construct multi stage systems to overcome these limitations. Considering a
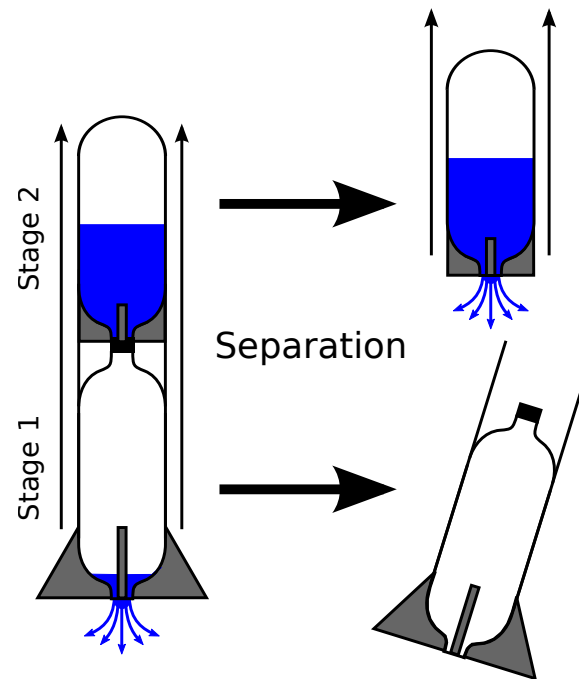
Figure 2.5.: Two-stage water rocket

chemical rocket, it is only logical to drop a certain empty volume after the previously contained fuel has been ignited. After that, the rocket has to carry less weight and is therefore more efficient. For water rockets the staging seems even more logical. Equation (2.9) indicates that the exhaust velocity and thereby the thrust is highly dependent on the pressure. Thus, more stages will not only result in higher efficiency by dropping unnecessary weight, but also provide more phases of high exhaust velocity. This demonstrates the importance of a multi-stage scheme for the optimisation of water rockets. For instance, one could try to implement many small stages instead of one or two larger ones to increase the thrust phases. In chapter IV, this and some other optimisation ideas will be discussed in more detail.

Fortunately, no additional equations are needed to describe more stages. A multi stage system can simply be viewed as an addition of several single stage rockets. Every one passes its height and velocity at the separation point to the next stage. All other stages above the active one will act as additional payload. Therefore, an implementation of multi-stages is a rather simple task, which will be presented in the next chapter along with the simulation of single stage rocket and the specific optimisation methods.

III. Numerical implementation

3.1. Flight diagnostics

Before optimising the rockets parameter a so called "full-output" program is needed. This should solve the equations (2.15) - (2.21) and write out diagnostics. The functionality of the program will be described in this section with an exemplary test run at the end.
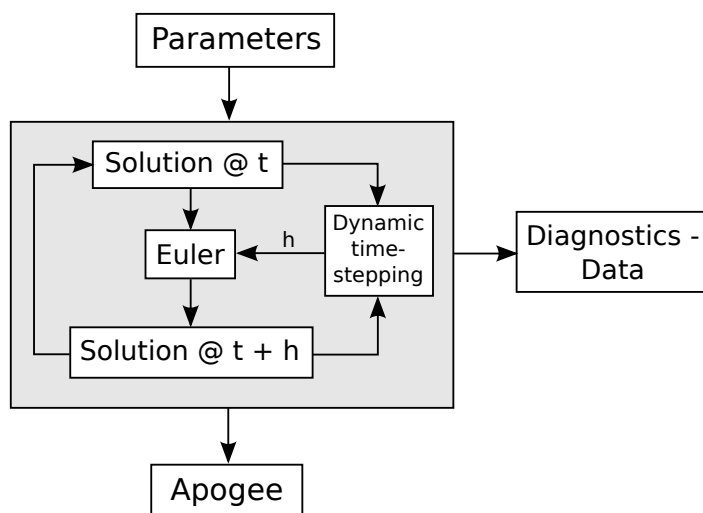
3.1.1. Functionality

Figure 3.1.: Flow chart of the single-stage program

Figure 3.1 shows how the program works in general. At first, the parameters of the rocket have to be determined. With these the initial values for the solution can be calculated. All further values are obtained by a stepwise temporal integration of the equations (2.15) - (2.21). At every step the values of time, pressure, rocket mass, rocket velocity, exhaust velocity, height and thrust are written out for later analysis. The integrator used for solving the differential equations is an explicit Euler algorithm [14].

$$y_i(x_0 + h) = y_i(x_0) + h \cdot y_i'(x_0) \tag{3.1}$$

Thus, the program approximates the functions $y_i(x)$ in small time steps h with straight lines. Of course, this algorithm is less accurate than higher order schemes

like a 4th order Runge-Kutta method, but since the equations to solve are not too complex this rather simple numerical approach seems adequate.

The Euler algorithm's accuracy is highly dependent on the chosen time step. For example, if a very small initial amount of water is inside the rocket, the emptying may happen in one single numerical step, which produces unrealistic high thrusts. For this reason a dynamic time-stepping is included into the code. It checks the temporal change of the variables and adjusts the time steps until these gradients are within reasonable intervals. Unfortunately, the drag term destroys conservation of energy and therefore such validation was not possible.

3.1.2. Test run

After all modules have been included in the program a first test will show its capabilities. The following characteristic water rocket parameters have been chosen:

$$
\begin{aligned}
\text{Initial internal pressure:} & \quad p_0 && = 5 \cdot 101325 \text{ Pa} \\
\text{Total rocket volume:} & \quad V_0 && = 1.5 \text{ l} \\
\text{Initial water volume:} & \quad V_W && = 0.75 \text{ l} \\
\text{Empty weight rocket:} & \quad m_0 && = 300 \text{ g} \\
\text{Nozzle radius:} & \quad r_N && = 1 \text{ cm} \\
\text{Bottle radius:} & \quad r_B && = 4 \text{ cm} \\
\text{Drag coefficient:} & \quad C_D && = 0.5
\end{aligned}
$$

With these parameters the simulated rocket reaches an apogee of 24.3 m, in agreement with experimental data [15]. Figure 3.2 shows the rocket's height and velocity as a function of time with and without air thrust. The actual thrust phase, in which the rocket accelerates, lasts less than two tenths of a second. Afterwards the flight is similar to a basic parabolic trajectory, since only drag and gravitation are acting on the rocket. Therefore, the more interesting part within the first tenths of a second is plotted in the figures 3.3 and 3.4. The air thrust term is always included here.

In figure 3.3 the two thrust phases can be easily identified. At 0.125 s the air thrust phase starts, which is much shorter and has a significantly higher velocity than the water phase before. But since the exhausted mass is also a lot smaller (see figure 3.4) the thrust is rather low compared to the one with water. The impact of the air thrust on the trajectory is shown in figure 3.2. The dashed lines represent a program run in which the remaining pressurised air is just neglected. Obviously, the effect on the maximum velocity is small, but still affects the rocket to fly about seven percent higher and therefore can not be ignored.

Figure 3.4 also shows an error within the program. While the pressure is already down to atmospheric level the mass has not returned to the empty rocket mass plus air mass inside. The reason for this must be the not fully consistent description of the gas outflow, but is due to its smallness acceptable.
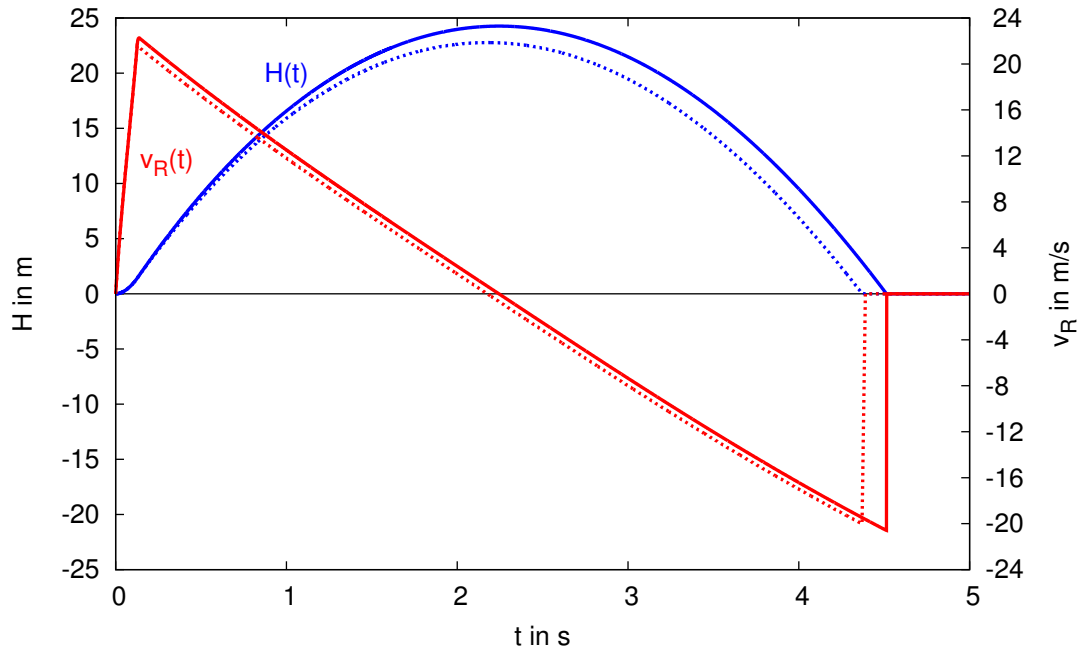
Figure 3.2.: Test run: Height (blue) and velocity (red), with (solid) and without (dashed) air thrust
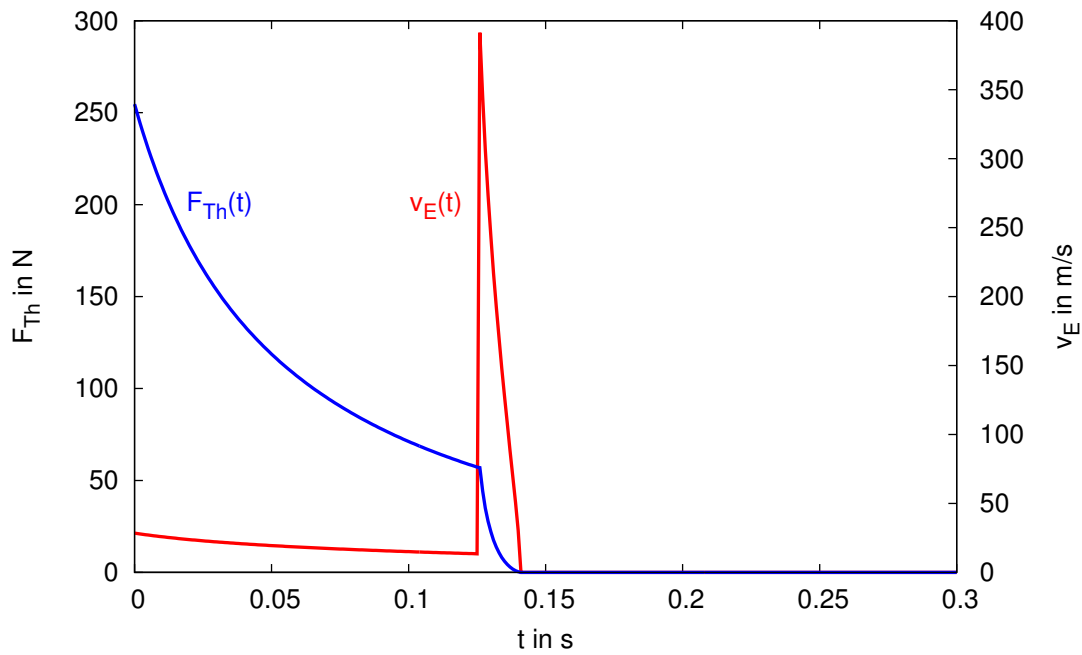

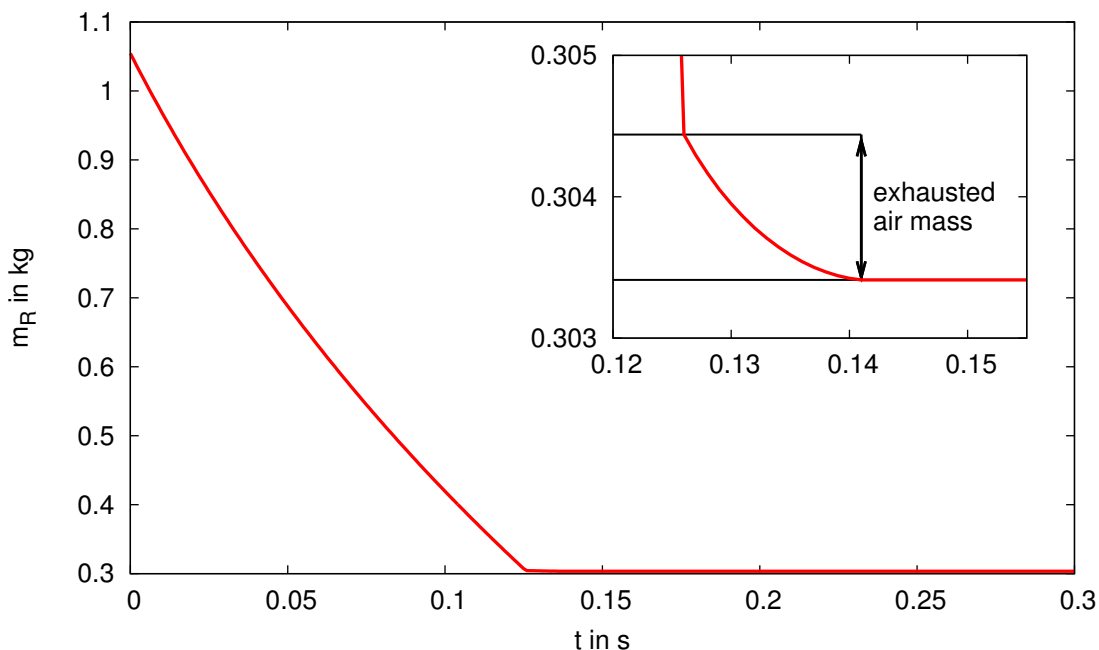Figure 3.3.: Test run: Thrust (blue) and exhaust velocity (red)

Figure 3.4.: Test run: Total rocket mass

3.1.3. Multi-staging

After having a working single-stage simulation the next step is to implement one or more additional stages. Those act like independent single stages with additional payload. So, the multi-stage program works as shown in figure 3.5. The box "Rocket simulation" represents the grey box in figure 3.1. As one can see the first stage gains extra weight from the stage above. The second stage is just a second call of the single-stage program with non-zero initial values for height and velocity. Those are given by the final values of the first stage and therefore depend on the time of stage-separation. The program can easily be extended by additional stages since only further program calls are needed.

3.2. Optimisation

In order to optimise the water rocket an algorithm is needed, which varies the input parameters. The apogee reached by the rocket will be the parameter to be optimized. In this section the applied algorithms will be introduced while the discussion about the parameters will take place in the next chapter.

3.2.1. Monte Carlo Method

As described by Dickman and Gilman [16] Monte Carlo can be very easily used for optimisation in a statistical approach. One searches for an optimum by generating
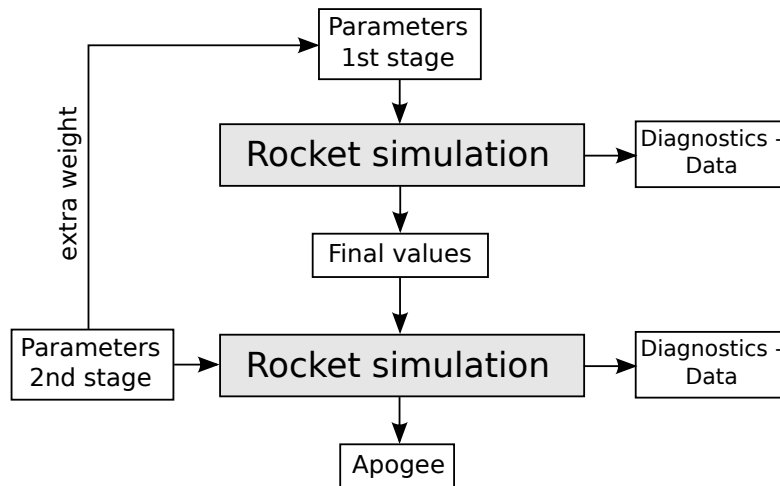
Figure 3.5.: Flow chart of the two-stage program

many solutions (about one million). The parameters for these solutions are randomly chosen within certain intervals. It can be improved by a multi-level algorithm, which uses the obtained optimum as the new central point for narrower intervals of the parameters. By repeating this one obtains the numerical optimum with high accuracy. Disadvantages of this method in contrast to deterministic algorithms like Newton or Gradient Descent are the much higher computing time and the fact, that there is no information about the distance to the true optimum. An advantage is, that a Monte Carlo scan not only delivers an optimum, but also provides a description of the solutions topology. Therefore, this method was chosen as the main optimisation tool.

3.2.2. Gradient Ascent

To check the Monte Carlo optimisation procedure introduced before an alternative algorithm was also included into the program [17]. Usually it is known as Gradient descent method, but since the searched optimum is the maximum height of the rocket, the name Gradient Ascent seems more adequate. The algorithm starts at an arbitrary parameter point and then calculates the gradients in every parameter direction (forwards and backwards), while the other parameters are kept constant. After that, the direction with the highest ascent is chosen as the direction into which the search is continued. If no positive ascent is found the step length will be shortened. The algorithm is repeated until at least one direction shows zero ascent, while all others are negative. This point represents the optimum.
Of course, the algorithm could also finish in a local maximum failing to find the global one. Here, a combination of the two methods is very helpful. With the topology of the parameter space known from the Monte Carlo scan it is easier to decide whether the maximum found by Gradient Ascent is global or local. Even a method is possible, in which a Monte Carlo algorithm randomly chooses the starting points

for the Gradient Ascent method. In this particular problem such a hybrid is not necessary.
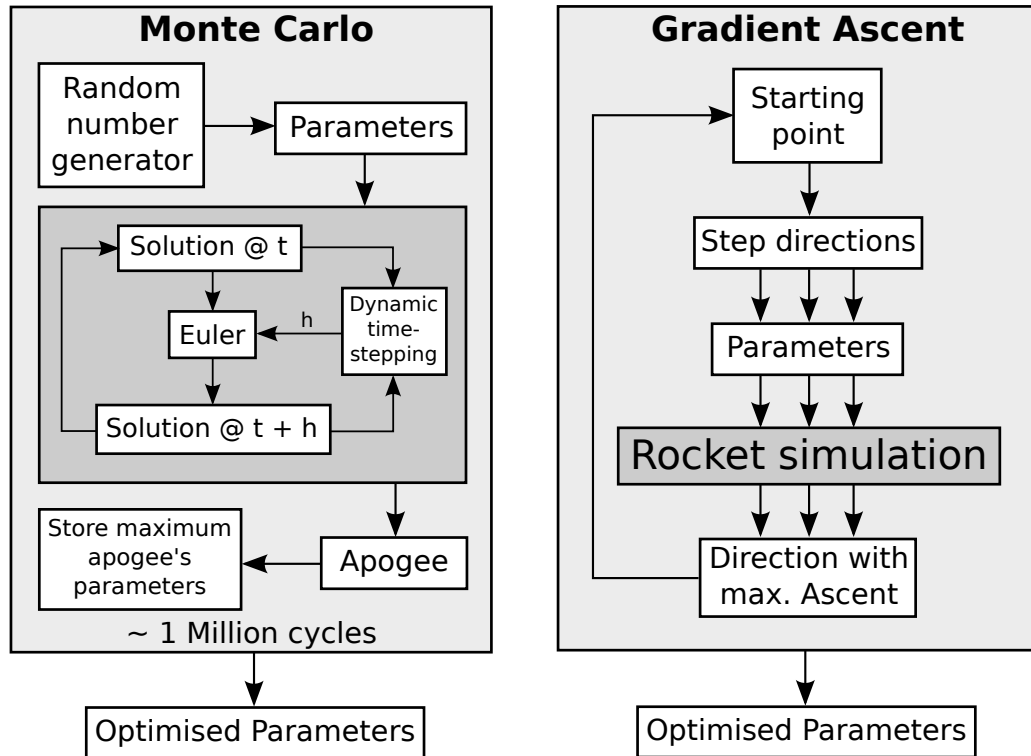
3.2.3. Enhanced program

Figure 3.6.: Flow chart of the optimisation programs

The implementation of both optimisation methods is shown in figure 3.6. For Monte Carlo there was only a random generator added and an algorithm, which compares the results and stores the parameter set with maximum apogee. The Gradient Ascent on the other hand is a little bit more complex, since there has to be a program call for every direction in the parameter space and an algorithm which evaluates each direction to get a new starting point. The box "Rocket simulation" in the right flow chart represents the box with the same color in the left chart. After finishing, both methods return the parameters set with the highest apogee. Monte Carlo also saves all parameters sets and the corresponding height for additional analysis of the parameter space topology. The performance of both algorithms will be discussed in the following chapter.

IV. Results

4.1. Parameters

In this section the parameters of single and multi-stage water rockets will be evaluated in view of their optimisation potential. These parameters are:

$$
\begin{aligned}
\text{Initial internal pressure:} \quad & p_0 \\
\text{Empty rocket mass:} \quad & m_0 \\
\text{Nozzle radius:} \quad & r_N \\
\text{Bottle radius:} \quad & r_B \\
\text{Drag coefficient:} \quad & C_D \\
\text{Total rocket volume:} \quad & V_0 \\
\text{Initial water volume:} \quad & V_W \\
\text{Separation time:} \quad & t_{sep}
\end{aligned}
$$

Drag coefficient, empty rocket mass, bottle radius and total rocket volume can be ruled out. The purpose of this optimisation is to find ideal parameters for a rocket with given volume, dimension and empty weight. That leads to the conclusion, that their optimisation is rather pointless. It is also obvious, that a smaller drag coefficient is always better, since it scales the strength of the inhibiting drag. Therefore, its optimisation is more of an engineering problem and the parameter is seen as given. Equations (2.16) and (2.19) indicate that a higher pressure difference leads to more exhaust velocity. The only negative effect is the increasing air mass. This is negligible due to the very low density of air. Limitations for the pressure are only given by the rocket's material. So, in all further optimisation runs a typical pressure will be applied. The remaining parameters will be discussed in more detail now.

If not specified otherwise, the parameters for all separate stages are the same as in subsection 3.1.2.

4.1.1. Nozzle radius

By looking at the model it is not clear what influence the radius of the nozzle has on the rocket's performance. Obviously, a higher radius increases the time derivative of the mass, what results in more thrust. On the other hand, it also makes the bottle to empty itself faster, cutting the time span of thrust. That leaves two scenarios:

either an optimum nozzle radius exists or a higher radius always increases the apogee. A Monte Carlo scan over different nozzle radii, varied between zero and the bottle radius, revealed the curve in figure 4.1.
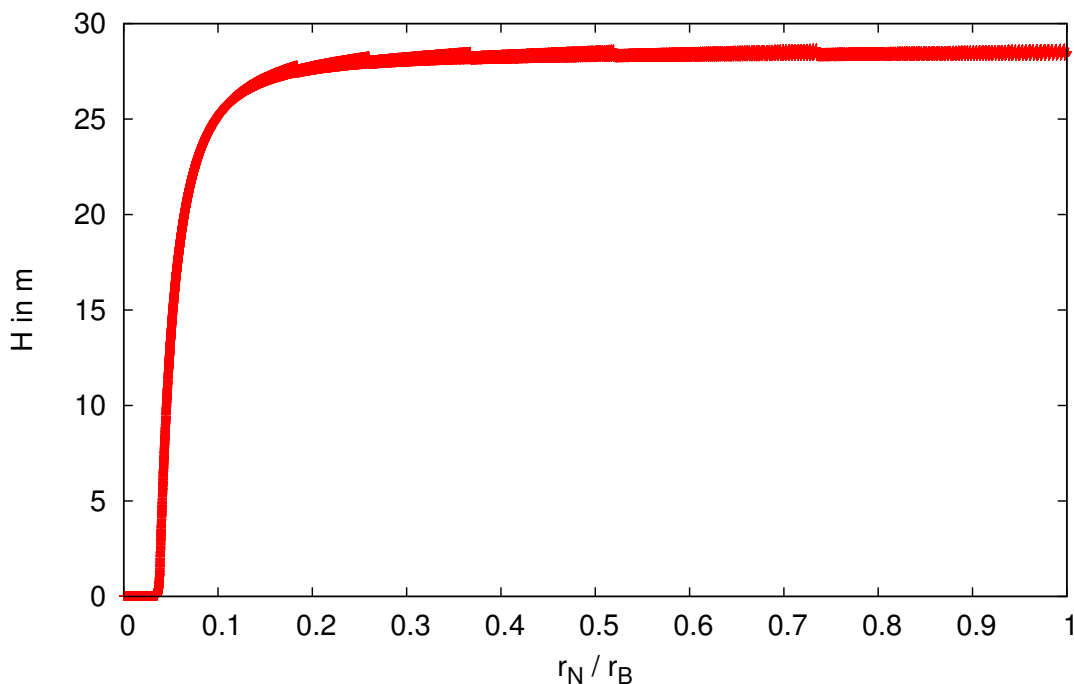

Figure 4.1.: Monte Carlo scan of the nozzle radius

As one can see, there is no optimum within this parameter space. Also, while getting closer to the bottle radius, the assumption of a slow internal water movement compared to high exhaust velocity becomes invalid. Therefore, the whole model is no longer valid at higher nozzle radii. Maybe, by applying a more complex fluid model, an optimum radius could be obtained. But within the model of this work it is not possible to find one. Fortunately, the common nozzle radius of water rockets is about one centimetre (water bottle) or even less (so called gardena nozzles [18]). Therefore, this parameter will hereafter also be treated as given.

It may attract attention that the plot has some kind of a saw-tooth form with a certain width. This problem will also occur in further scans and will later be analysed in more detail.

4.1.2. Separation time

The separation time is only applying for multi-stage rockets. It defines the point in time when the lower stage is released. Two possible scenarios are plausible, namely to separate at maximum rocket velocity or at apogee. Typical chemical rockets separate when the fuel inside the current stage is fully ignited, what favours the first option. In

addition, earlier dropping of unnecessary mass results in a higher rocket momentum, increasing the apogee. So theoretically, the dropping should happen at the time of burn-out. The optimisation algorithms introduced before allow to validate this.
The maximum height's dependence on the separation time, is shown in figure 4.2 (blue). The red line shows the rocket velocity without a happening separation.
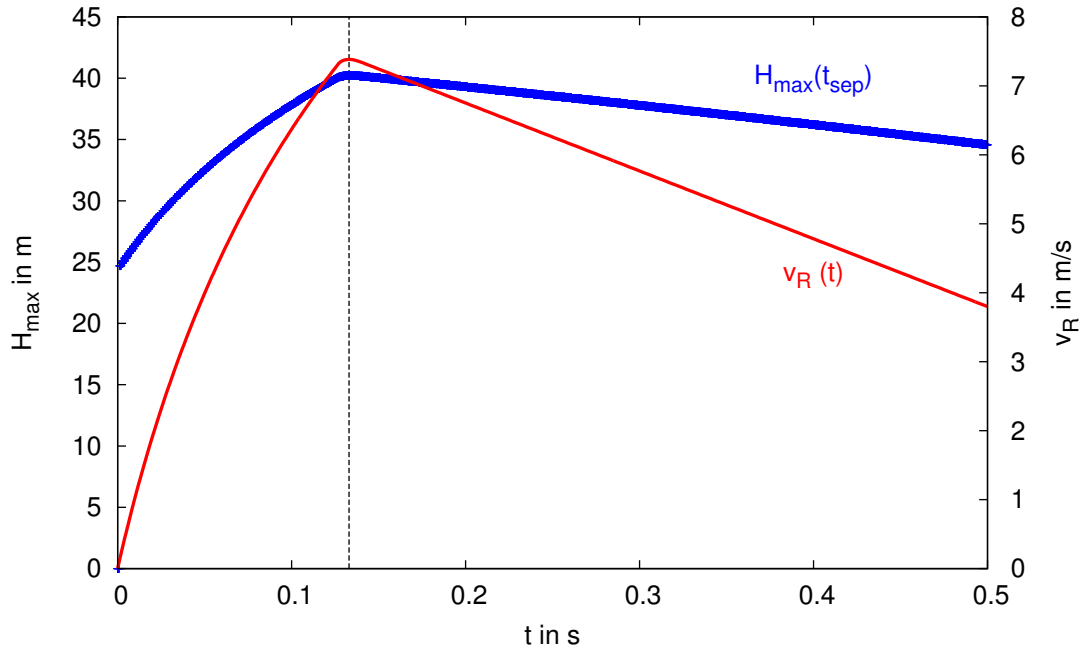

Figure 4.2.: Monte Carlo scan of separation time (blue) and velocity of the first stage (red)

The dashed line helps to see that the optimised separation time and maximum velocity occur at the same time. So, this first optimisation copes with the theory and therefore also represents a validation of the applied Monte Carlo algorithm. As a result, in all further program runs, the stages will separate at the time of maximum velocity of the current stage.

4.1.3. Initial water volume

The last free parameter of a water rocket is the initial water volume inside the bottle or, to put it into a more general form, the ratio between water and air volume $V_{W,0}/V_{A,0}$. When searching on the internet one finds many different values between one-third and one-half. The reason for this fluctuation will be investigated in the next section. Further simulations showed that the ratio between air and water is one important parameter for water rockets, maybe the most important one. Even small changes lead to significant differences in the maximum flight height. When building multi-stage rockets it is even more dominant, since not all stages have the same optimum ratio.

It becomes apparent that the ratio between water and air is the crucial factor for optimisation. Therefore, the next two chapters will show the result of optimising this parameter for single and double stage rockets. Subsequent to this, the last section will describe more exotic optimisation models to demonstrate the adaptivity of the simulation.

4.2. Single-stage optimisation
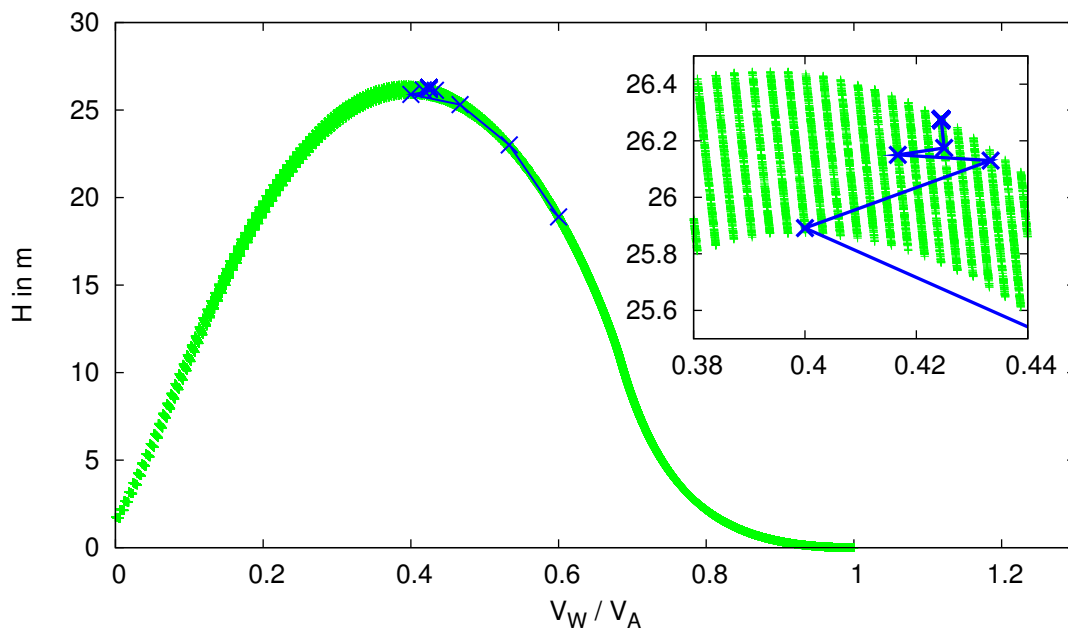
4.2.1. Monte Carlo and Gradient Ascent

Figure 4.3.: Monte Carlo (green) and Gradient Ascent (blue) optimisation

For Monte Carlo optimisation one million random fuel ratios $V_{W,0}/V_{A,0}$ between zero and one have been prepared as input for the program. The starting point of the Gradient Ascent algorithm was chosen as two thirds. After finishing both optimisations the data has been plotted into one chart (figure 4.3) to show the difference between the methods. The results are:

$$\begin{aligned} \text{Monte Carlo:} \quad & V_{W,0}/V_{A,0} = 0.3938 \quad & H_{max} = 26.4467 \text{ m} \\ \text{Gradient Ascent:} \quad & V_{W,0}/V_{A,0} = 0,4245 \quad & H_{max} = 26.2758 \text{ m} \end{aligned}$$

As already suspected the Gradient method is not able to find the same maximum like Monte Carlo. Looking on the magnified section of the plot, it stands out that the function has a certain distribution width. This is not an error of the Monte Carlo

scan, but of the integrator. This becomes more obvious in figure 4.4. The right picture shows a very close look on the apogee. Since the Euler method fills the gaps between two points with lines the maximum is determined by the time-steps. On the picture the highest point does not seem to be the highest point of an imaginary real parabola. Now every slight variation of the fuel ratio has the effect that the rocket can fly higher, but also longer. Therefore, the next higher apogee is shifted sideways to the current one. Of course, in periodic distances the found apogee will match the maximum of the parabola. But for the next ones, found with Monte Carlo, the error will continuously rise until another parameter set hits the real maximum. The result is a strip pattern that reveals when looking to the broadened Monte Carlo scan in the left picture.

Due to that the solution has many artificial local maximums. Unfortunately, the Gradient Ascent ends in one of them, when its stepsize becomes too small to reach the next maximum. The only solution for this would be to implement a higher order integrator. But since Monte Carlo is still producing very good results, it has been decided that the integrator is kept. The Gradient Ascent method, however, will not be further used, besides for cross-checking, where its speed turned out to be very helpful.
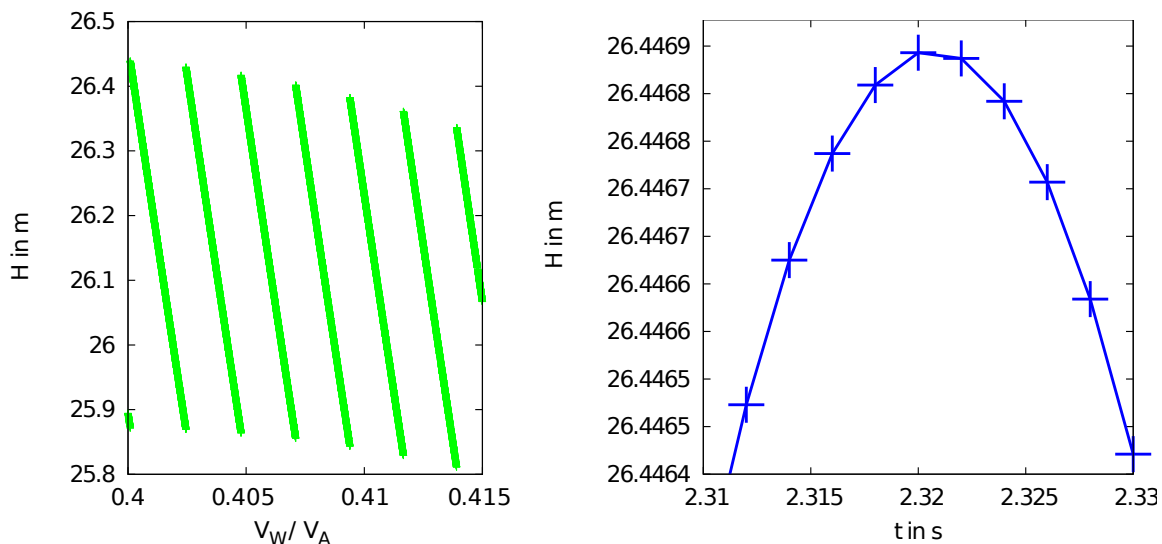

Figure 4.4.: Magnified Monte Carlo scan (left) and trajectory (right)

4.2.2. Pressure dependency

After achieving an optimum of about two-fifths the question remains, why the values found in the internet differ so much. After some further simulations it became obvious that the ratio is very dependent on other parameter, which have been excluded from the optimisation before. Because the pressure is a parameter easy to vary, it was chosen for another optimisation. For this the Monte Carlo program was slightly

changed. A second Monte Carlo loop was combined with the existing one. While
the first loop searches for optimum fuel ratios, the second one will vary the internal
pressure. By this, one obtains a chart of optimum fuel ratio over pressure. The
result can be seen in figure 4.5. As expected, the fuel ratio shows a clear pressure
dependency. The small line of points in the upper left of the graph is within a region,
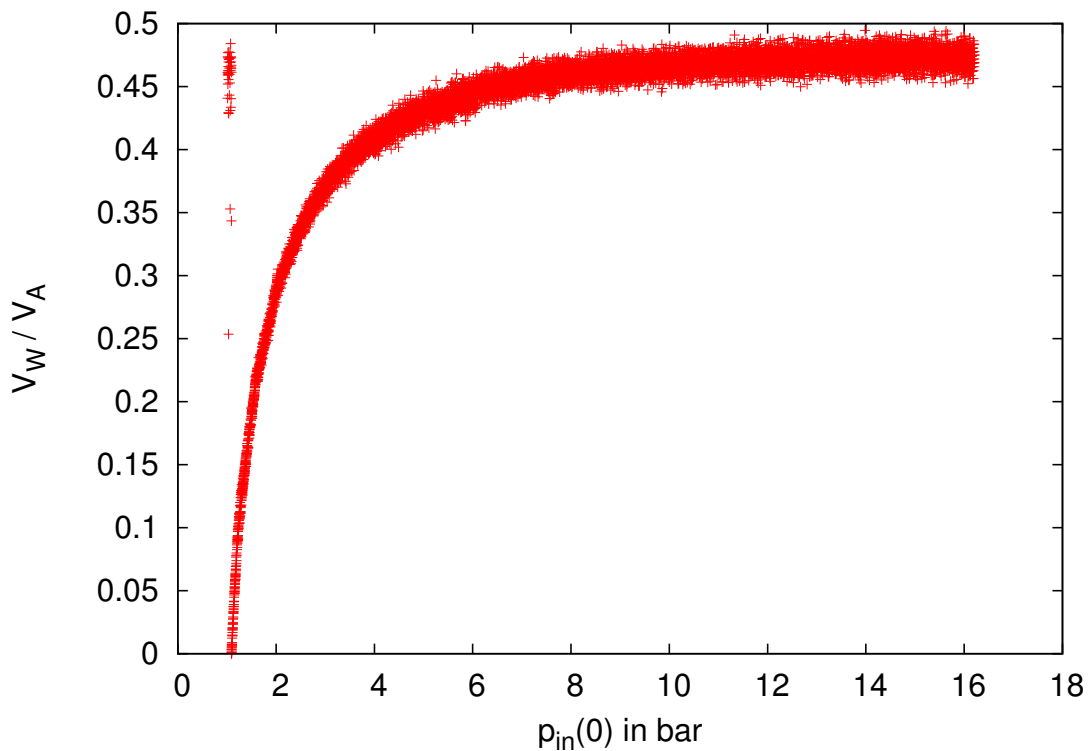where the rocket reaches almost no height and can be ignored therefore.

Figure 4.5.: Pressure dependent optimum fuel ratio

Unfortunately, the graph is rather sparse, due to the fact that two Monte Carlo
algorithm had to be run simultaneously. In order to contain the calculation time
within manageable limits it was necessary to reduce the number of loops. Therefore,
the real maxima could not have been determined as accurate as in the previous
runs. In numbers, only one thousand fuel ratios for each Monte Carlo run and
ten thousand pressures have been applied to the optimisation. Compared to the
one million fuel ratios used before, this rather small number explains the broadened
graph. Fortunately, the results are good enough to identify the functional trend
of the dependency. However, further calculations should use much higher numbers
of parameter sets while compensating the additional calculation time with parallel
programming.

4.3. Two-stage optimisation

4.3.1. Monte Carlo

After the conclusion that the Monte Carlo method is clearly superior to Gradient Ascent, the optimisation of the two-stage rocket is done only with Monte Carlo. In addition to the single stage optimisation a second fuel ratio is needed, so both volumes can be optimised separately. The optimised ratios are:

First stage: $V_{W,0}/V_{A,0} = 0.4577$
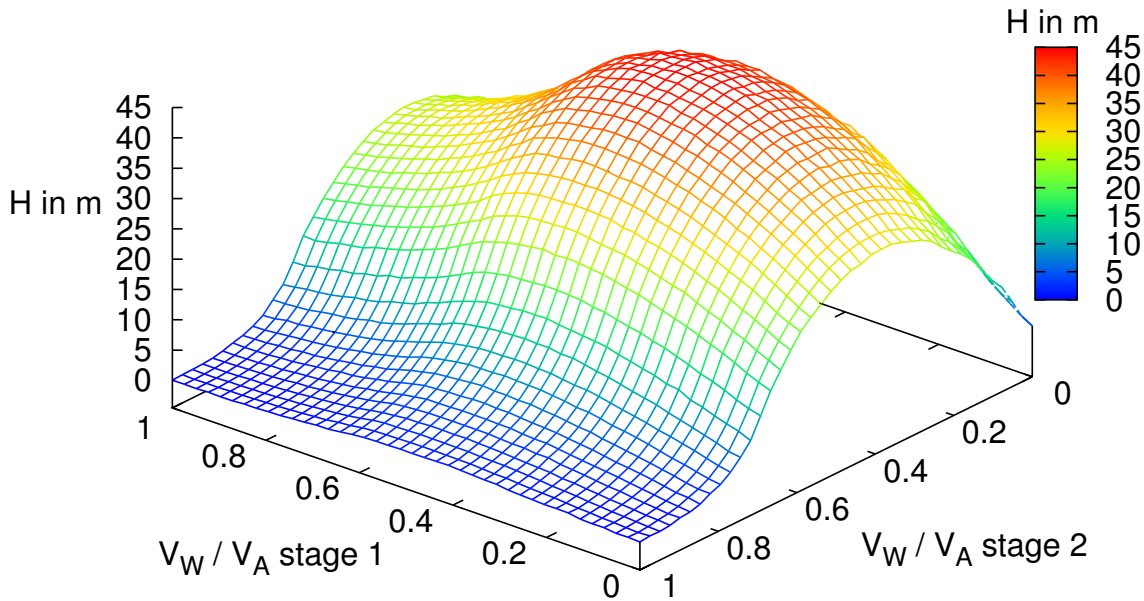Second stage: $V_{W,0}/V_{A,0} = 0.3423$

Figure 4.6.: Two-stage rocket: Monte Carlo optimisation

With these ratios, the simulated two-stage rocket was able to reach an apogee of 45.3973 m. Clearly this is a big advantage comparing to the single-stage rocket. It is very interesting, that both rockets have different optimum ratios. For a better visualisation the topology of the parameter space in figure 4.6 is plotted in three dimension. Obviously, there is a rather complex functional dependency between height and both fuel ratios. The fact, that the flight height goes down to zero, while the second fuel ratio goes toward one, is due to the applied program. Even though the second stage would not produce enough thrust for a positive velocity, it still can reach a certain height by the first stage. But since this optimisation is for two-stage rockets all results with no starting second stage are threaded as "no height reached".

4.3.2. Pressure dependency

As before it is very likely that this fuel optimum is only valid for one particular parameter set. Therefore, the introduced program for determining the pressure dependency

was applied for the two-stage model as well. The result is shown in figure 4.7, revealing two conclusions. First, the optimal ratio is, as expected, indeed pressure dependent. Second, the ratios for both stages are not only different but also have varying pressure dependency. This conclusion arises from the different ascent of both curves. Above six bar the optimum ratio of the second stage is not changing very much while the curve for the first stage is still rising.

As already stated in the single-stage optimisation section the broadening of the plots are due to the inaccurate Monte Carlo scan. For the two-stage simulation another parameter had to be varied and therefore even more parameter sets would have been needed. But since the used number of Monte Carlo runs is the same as before the results are even more inaccurate. To obtain a thinner curve much more calculation time would be needed, which was unfortunately not available. But like in the previous section the presented results are also good enough to see a clear trend.
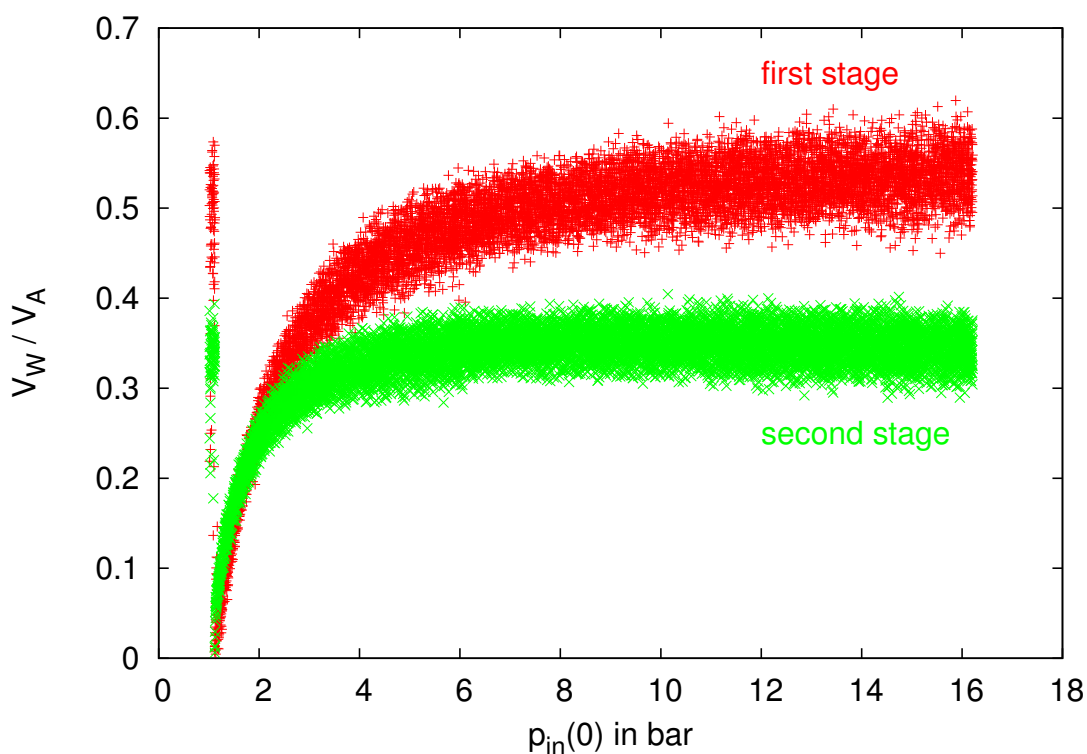

Figure 4.7.: Two-stage rocket: Pressure dependent optimum fuel ratio

4.4. Further simulations

In this final section the existing programs are used to simulate two unusual water rocket concepts in order to obtain further insights.

4.4.1. Aluminium rocket

It has already been discovered that a higher pressure corresponds to a higher apogee. Since standard water bottles are very limited in their pressure resistance one comes to the idea of using a different material. In medical applications small aluminium bottles with a capability of two litres are commonly in operation. These are used to contain oxygen with a charging pressure of two hundred bar, which is an obvious advantage compared to the maximum of 8 to 15 bar of a plastic bottle. On the other hand, such a bottle with 1.9 kg weights much more. After optimising a rocket with such parameters the simulation reached an impressive height of 222.8902 metres with an optimum fuel ratio of $V_{W,0}/V_{A,0} = 0.5476$. Interestingly, this optimum is clearly over one-half, while previous optima mostly stayed below this value.
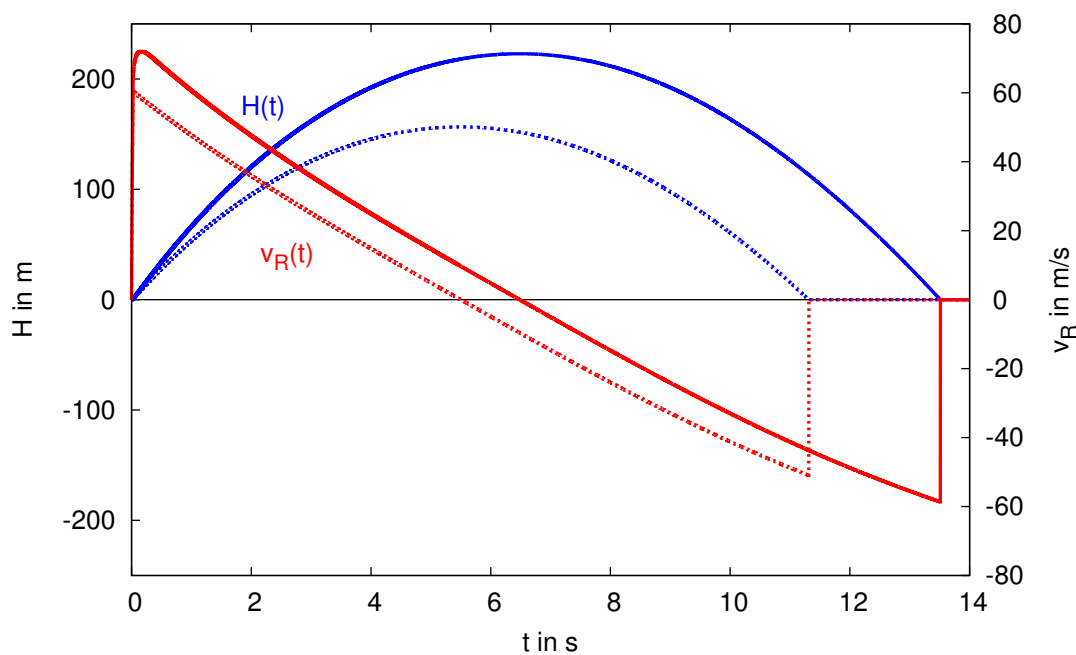

Figure 4.8.: Aluminium rocket: Height (blue) and velocity (red), with (solid) and without (dashed) air thrust

The trajectory of the flight is shown in figure 4.8. For comparison, the program run without considering the air thrust is plotted with a dashed line. Since the water is pressed out very quick, there is still very high pressurised air inside the bottle. This leads to much more air-thrust and a height difference of about 50 meters. Therefore,

an experiment with such an aluminium bottle should be a good validation of the applied air-thrust model. The simulation showed that an optimisation of material is a task with very promising results and should be further investigated. On the other hand, it should not be forgotten that a bursting aluminium bottle represents a much higher risk than a plastic bottle. For this reason, such experiments should be done with very high caution.

4.4.2. Multi-staging performance

The simulations revealed that multi-staging is, as already speculated in the first chapter, a very effective way to improve water rockets. But until now only further stages and therefore extra fuel was added to the rocket. To have a systematic analysis of the multi-staging concept, simulations were done with one-, three- and ten-stage rockets. But this time the overall amount of water was kept constant. So this is analogous to splitting a single water bottle into more parts instead of using a design with more bottles above.

This idea has been already introduced in section 2.4. There it has been presumed that these additional phases of high exhaust velocity would increase the performance. So the parameters for these three simulations are overall the same, with the exceptions that the water volume and the weight is split into three respective ten parts for the multi-stage rockets.
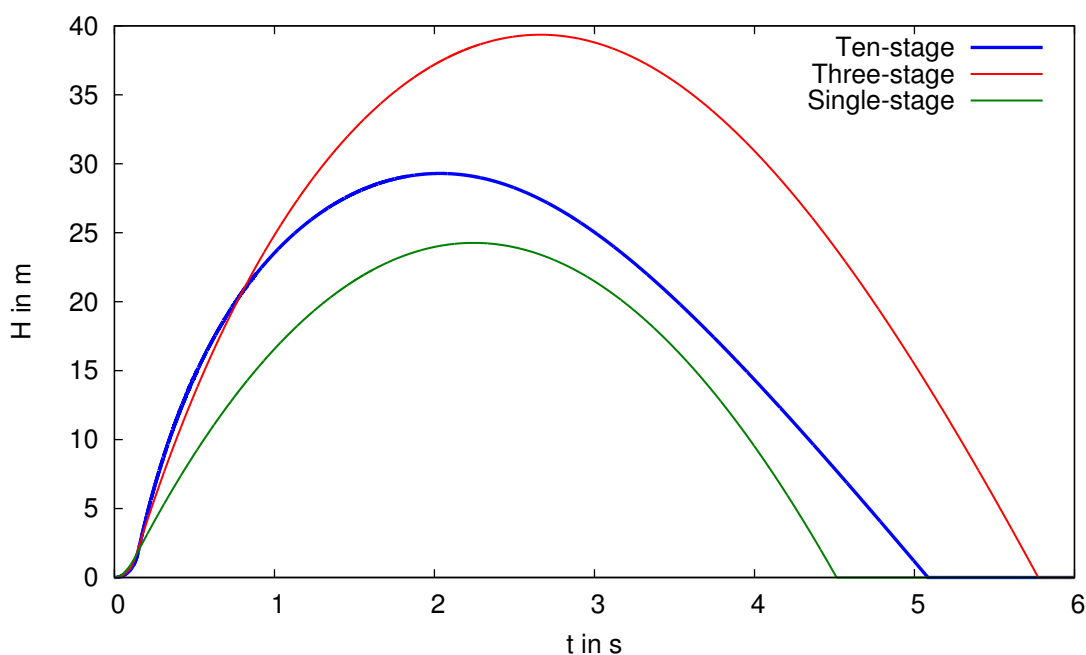

Figure 4.9.: Staging comparison: Height

The results in figure 4.9 and 4.10 are quite unexpected. Apparently, the rocket with

ten stages reaches a lower apogee than the one with three stages. Figure 4.10 reveals
that the ten-stager is not able to hold its velocity as long as the other two rockets.
This actually seems logical, since it drops more weight and therefore has a lower
impulse. This impulse then approaches faster zero, being decreased by drag and
gravitation, what leads to an earlier and also lower apogee. This shows that more
stages, while keeping the same overall amount of fuel, do not necessarily correspond
to a higher apogee. On the other hand, the ten-stage rocket reaches its apogee
much faster than the others. So, if a very fast ascent is wanted, this rocket is more
adequate. It can be summarised that the number of stages has to be adapted to the
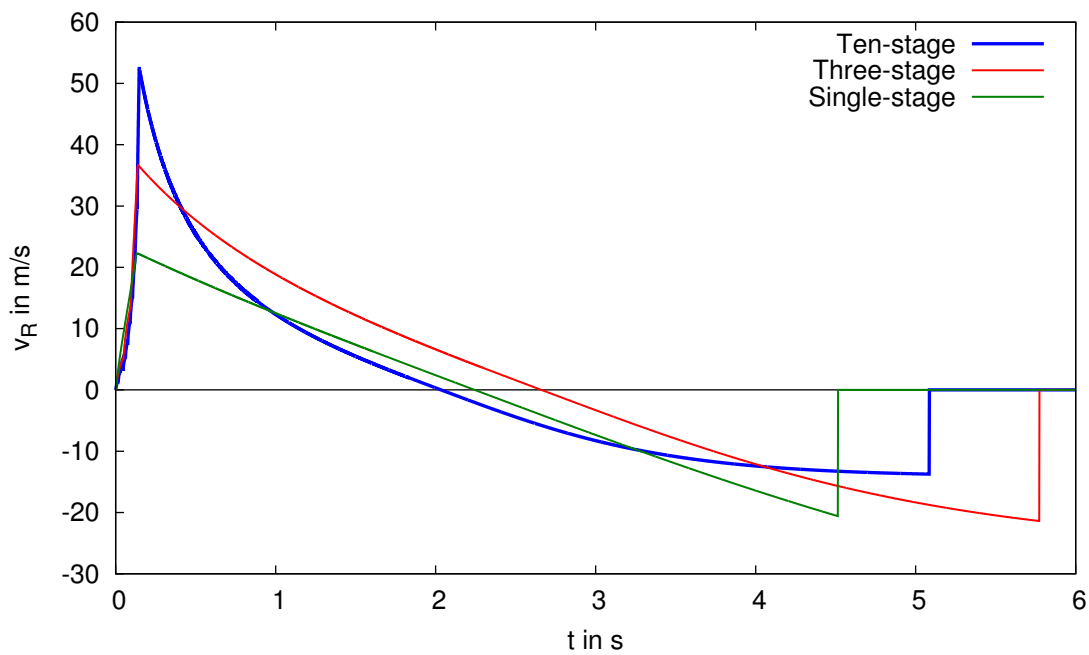purpose and properties of the specific rocket like other parameters.

Figure 4.10.: Staging comparison: Rocket velocity

V. Conclusions and outlook

This thesis showed that it is possible to simulate and optimise water rockets with a numerical approach. In the simulations the flight behaviour of such rockets and the influence of the frequently neglected air-thrust became obvious. The potential of multi-staging emerged also very well. Even though the applied model neglects some effects like crosswind effects or internal water flow the results match with experiments. Unfortunately, precise and comprehensible data is very rare due to the toy-like appearance of water rockets. Therefore, it would be necessary to validate the modelling results with specific experiments in the future. From a computational point of view, it seems possible to improve the results with a more accurate integrator. By applying a more complex fluid model, crosswind and precise outflow could enhance the model as well.

The optimisation revealed that there is no robust optimum ratio between water and air, because all other parameters have a certain influence on it. It was possible to analyse this dependence in case of internal pressure, both for single- and two-stage rockets. As a further step, the calculations of the pressure dependent optimum fuel ratio should be done with more accuracy and for different rocket volumes. By this, it might be possible to get a reliable prediction what ratio should be used for a specific rocket set-up. Here, experimental validation is also needed.

The numerical study of unusual designs of water rockets at the end demonstrated the adaptivity of the model very well. It was quite surprising how high an aluminium rocket should be able to fly, just by water and pressurised air, demonstrating the potential for optimisation by advanced materials.

In summary, this thesis has accomplished its goal to simulate and investigate the behaviour of water rockets. I hope it will encourage others to build and improve water rockets, because the joy of watching a self made water rocket reaching the sky can never be described in words, numbers or charts.

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