Quantum dynamics of a spin-boson system close to a classical phase transition

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1 Introduction

With the continuous rise of quantum engineering in solid state devices, well known models of atom-field interaction come into focus once again. The most generic of these, the Rabi model, implements the physics of competing timescales in a very clear way. In particular, it features a classical phase transition whose influence on the quantum mechanical behaviour are observable in the ground state and dynamical properties.

1.1 Rabi model

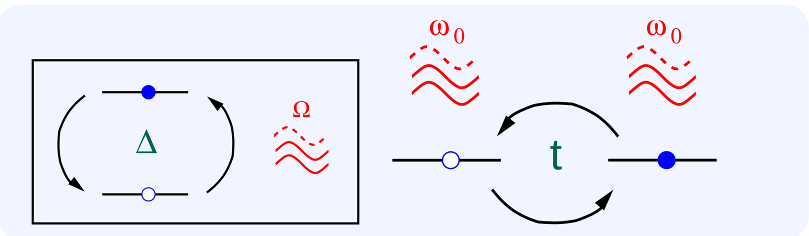
- Spin 1/2 + harmonic oscillator

$$H = \frac{\Delta}{2}\sigma_x + \epsilon\sigma_z + \Omega b^\dagger b + \gamma\sqrt{\Omega}(b^\dagger + b)\sigma_z$$

Δ: spin frequency, Ω: oscillator frequency, γ: coupling

- Realizations

Atom in cavity *Two-site Holstein Model*



- Note: interaction includes "counter-rotating" terms ($b^\dagger\sigma_+$, $b\sigma_-$ after spin rotation about $\frac{\pi}{2}$ around y-axis), in difference to the much simpler Jaynes-Cummings model of quantum optics.
- Reflection symmetry: H invariant under

$$\sigma_z \mapsto -\sigma_z\,; \quad b \mapsto -b$$

  - broken through external field $\epsilon \neq 0$
  - possibly broken in phase transition
- No analytical solution, but 'exact' numerics possible

1.2 Topics

(i) Classical phase transition at $\Omega = 0$ (static limit), but no "strict" quantum phase transition at $\Omega > 0$
How does the ground state properties evolve when the osc. frequency approaches the $\Omega \to 0$ limit?

(ii) "Fast oscillator limit" $\Omega \gg \Delta$
How does the ground state evolve at large coupling far away from the $\Omega = 0$ phase transition?

(iii) Nature of ground state
Is a simple variational ansatz able to describe the different ground state properties?

(iv) Renormalization of spin dynamics
How is the actual spin frequency linked with the renormalized $\widetilde{\Delta}$ in the ground state?

(v) Quantum dynamics
Can the ground state phase transition be recognized also in the dynamics?

2 Ground state properties

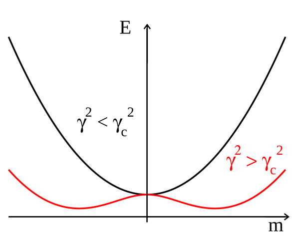
2.1 Exact limiting cases

Static oscillator limit: Phase transition

- At $\Omega = 0$ the model undergoes a phase transition: below the critical coupling $\gamma_c^2 = \Delta/2$ the ground state is non-degenerate with $\langle\sigma_z\rangle = 0$ (order parameter), above γ_c it is two-fold degenerate with $\langle\sigma_z\rangle \neq 0$.
- Integrating out the oscillator coordinates:

Landau functional for spin

$$E(m) = -\gamma^2 m^2 - \frac{\Delta}{2}\sqrt{1-m^2}$$



- Phase transition with mean-field exponents:

$$m = \langle\sigma_z\rangle = \begin{cases} 0 & \gamma < \gamma_c \\ \pm\sqrt{1-\frac{\Delta^2}{4\gamma^4}} & \gamma > \gamma_c \end{cases}$$

$$\chi = -\frac{\partial m}{\partial\epsilon} = \begin{cases} \frac{1}{2(\gamma_c^2-\gamma^2)} & \gamma < \gamma_c \\ \frac{\gamma_c^4}{2\gamma^2(\gamma^4-\gamma_c^4)} & \gamma > \gamma_c \end{cases}$$

- Ground state: $|\Psi_{static}\rangle = \frac{1}{\sqrt{2}}(\sqrt{1+m}\,|\uparrow\rangle + \sqrt{1-m}\,|\downarrow\rangle) \otimes |\alpha\rangle$
coherent state $b^\dagger|\alpha\rangle = \alpha|\alpha\rangle$

Fast oscillator limit: Spin renormalization

- $\Omega \longrightarrow \infty$, keeping $\gamma^2/\Omega = const.$
- Lang- Firsov transformation

$$U = e^{-\frac{\gamma}{\sqrt{\Omega}}(b-b^\dagger)\sigma_z}$$

maps on effective spin-model $H_{LF} = \widetilde{\Delta}\sigma_x$
"effective" spin frequency $\widetilde{\Delta} = \Delta e^{-2\gamma^2/\Omega}$

- Ground state: $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\alpha\rangle \pm |\downarrow\rangle \otimes |-\alpha\rangle)$

2.2 Small oscillator frequency

- Precursors of phase transition in quantum model
  - finite $\Omega \ll 1$, $\epsilon \to 0$
  - χ diverges at $\gamma = \gamma_c$ for $\Omega = 0$, almost divergent for $\Omega \to 0$

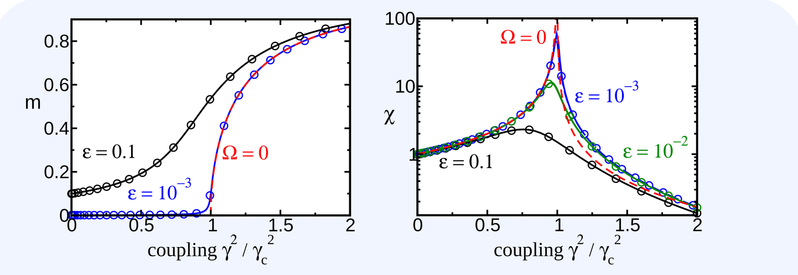


Fig. 3: Magnetization m (left) and susceptibility χ (right) as a function of coupling for frequency $\Omega = 10^{-3}$ and $\epsilon = 0.1$ (black), $\epsilon = 0.01$ (green) and $\epsilon = 0.001$ (blue). Circles: Results from the variational ansatz (see below).

2.3 Limit $\Omega \to 0$

- Oscillator behaviour:
  - $\gamma < \gamma_c$: oscillator can't follow spin for small Ω
  - $\gamma > \gamma_c$: crossover from spin-dependent shift at large Ω to global shift near $\Omega = 0$
- Spin behaviour:
  - separatrix ($\gamma = \gamma_c$) between different phases

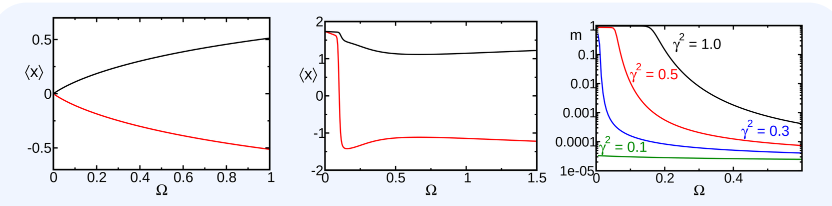


Fig. 4: Oscillator displacement $\langle x\rangle$ for $\gamma^2 = 0.16 < \gamma_c^2$ (left) and $\gamma^2 = 1.0 > \gamma_c^2 = 0.25$ (mid), Black: Spin up, Red: Spin down; Right: Magnetization m in the limit $\Omega \to 0$

2.4 Large oscillator frequency

- Renormalization emerges for $\Omega, \gamma \gg \Delta$
  - decrease of $\langle\sigma_x\rangle \sim e^{-2\gamma^2/\Omega}$ with increasing coupling
  - susceptibility increases as $\ln(\chi) = 2\gamma^2/\Omega$

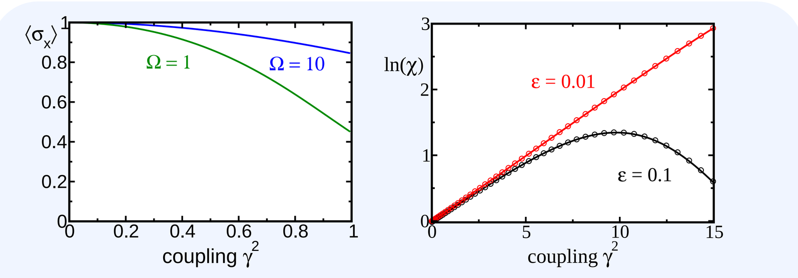


Fig. 5: Left: $|\langle\sigma_x\rangle|$ as a function of coupling for different oscillator frequencies; Right: $\ln(\chi)$ as a func. of coupl. for $\Omega = 10$ and finite ϵ; Circles: Results from the var. ansatz.

2.5 Ground state wave function & variational ansatz

- Small oscillator frequency
  - $\epsilon \approx 0$: one osc. state per spin below γ_c, two above; displacement is spin-dependent
  - $\epsilon > 0$: one osc. state with spin-independent shift

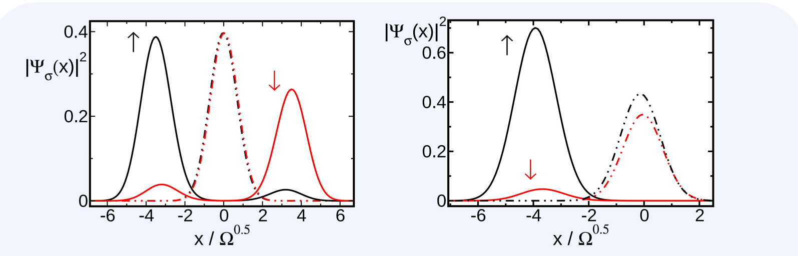


Fig. 6: Spin- projected wave functions for $\Omega = 0.1$ and $\epsilon = 10^{-5}$ (left) and $\epsilon = 0.1$ (right). Black: Spin up, Red: Spin down. Coupling $\gamma^2 = 0.01$ (dotted line) and $\gamma^2 = 0.5$ (solid line).

- Large oscillator frequency
  - $\epsilon \approx 0$: one osc. state per spin, symmetrically displaced
  - $\epsilon > 0$: also one osc. state per spin, but with different weights

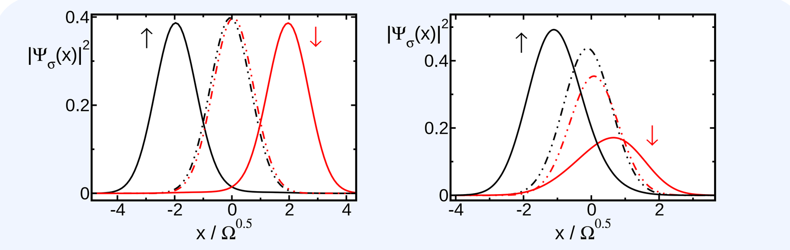


Fig. 7: Spin- projected wave functions for $\Omega = 2$ and $\epsilon = 10^{-5}$ (left) and $\epsilon = 0.1$ (right). Other parameters same as in Fig. 6

- Variational ansatz as mixture of coherent states:

$$|\Psi\rangle = a\,|\uparrow,\alpha+\beta\rangle + b\,|\uparrow,\alpha-\beta\rangle + c\,|\downarrow,\alpha+\beta\rangle + d\,|\downarrow,\alpha-\beta\rangle$$

  - $\epsilon = 0$: Symmetrization of $|\Psi_{static}\rangle$ ($\Omega = 0$)
  - $\epsilon \neq 0$: Ansatz allows for shift in spin position
- Excellent quality of ansatz: see Fig. 3

3 Dynamics

3.1 Large oscillator frequency

- Several phenomena can be classified according to the initial state and the time scale on which they occur

Renormalized Rabi oscillations
initial state $|\uparrow\rangle \otimes |-\gamma/\sqrt{\Omega}\rangle$; time scale $\widetilde{\Delta}^{-1}$

- Coupled Rabi oscillations of spin and oscillator
- Renormalization: Actual spin frequency equals effective spin frequency $\widetilde{\Delta}$ in the ground state

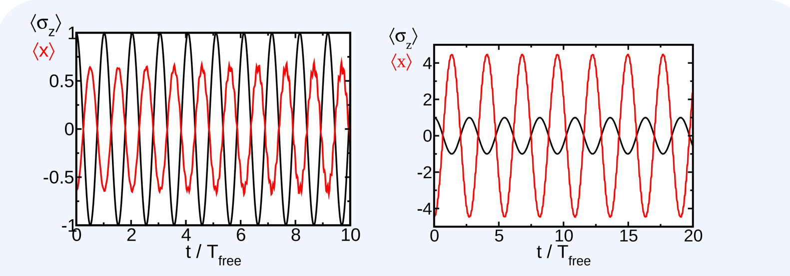


Fig. 8: $\Delta = 1, \Omega = 10$. $\langle\sigma_z\rangle_t$ (black) and $\langle x\rangle_t$ (red) propagating from the relaxed oscillator initial state with $\gamma^2 = 0.1$ (left) and $\gamma^2 = 5.0$ (right)

Collapse & Revival I
initial state $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes |\alpha\rangle$; time scale Ω^{-1}

- Oscillator propagates with bare freq.; collapse and revival of spin with revival time $T_{rev} = \frac{2\pi}{\Omega}$
- Behaviour fully determined by oscillator
- Explained by $\Delta = 0$ approximation:
  - Two coherent states oscillates independently; collapse-revival structure due to phase difference
  - Envelope $\times$ bare oscillation:

$$\langle\sigma_x\rangle = \exp[-4\frac{\gamma^2}{\Omega}(1-\cos(\Omega t))]\cos(4\frac{\gamma}{\sqrt{\Omega}}\alpha\sin(\Omega t))$$

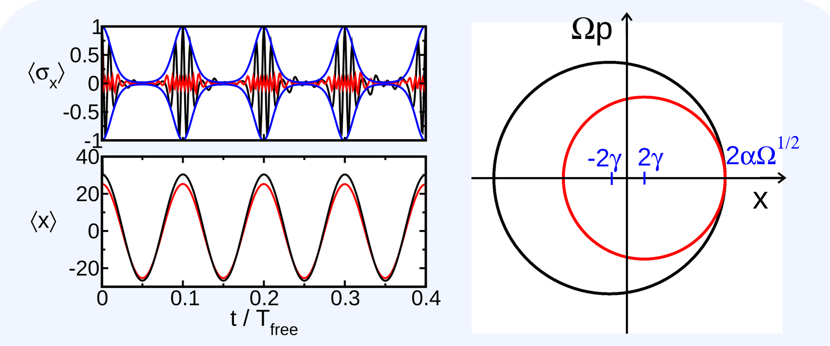


Fig. 9: $\Delta = 1, \Omega = 10$. Left: $\langle\sigma_x\rangle_t$ (above) and $\langle x\rangle_t$ (below) propagating from the coherent oscillator initial state with $\gamma^2 = 5$ and $\alpha = 10$ (black) and $\alpha = 1$ (red); Blue: envelope; Right: $\Delta = 0$ spin projected coherent states (black: spin up, red: spin down) propagating in phase space.

Collapse & Revival II
initial state $|\uparrow\rangle \otimes |\alpha\rangle$; large time scale

- Spin: collapse-revival with large T_{rev}; renormalized oscillation on short time scale
- Oscillator: beatings on the large time scale; bare oscillation on short time scale

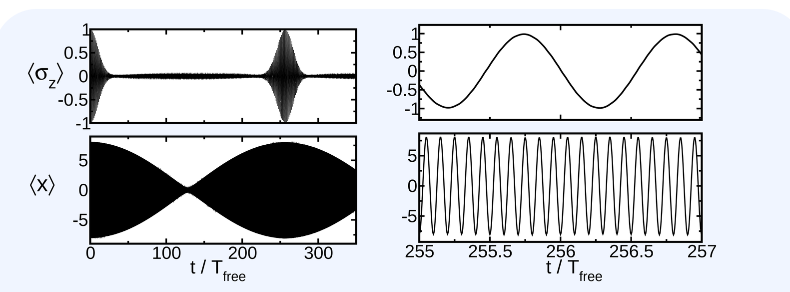


Fig. 10: $\langle\sigma_z\rangle_t$ (above) and $\langle x\rangle$ (below) propagating from the coherent oscillator initial state with $\Omega = 10$, $\gamma^2 = 0.01$ and $\alpha = 4$

- "Adiabatic approximation" for revival times:

$$T_{rev} = 2\pi/(\widetilde{\Delta}(L_{\alpha^2+1}(4\gamma^2/\Omega) - L_{\alpha^2}(4\gamma^2/\Omega))$$

$L_N(\cdot)$ ... Laguerre polynomials

  - Dynamics on large time scale is strongly dependent on renormalized spin frequency, coupling, as well as on the initial state parameter α.

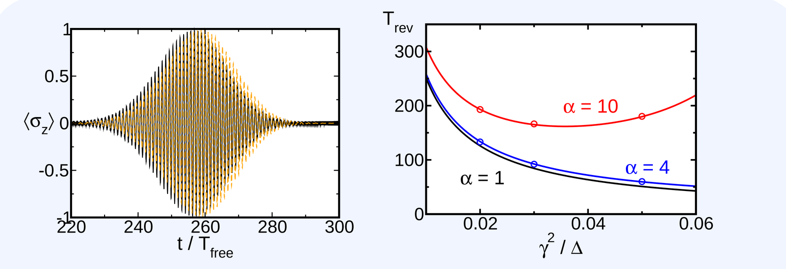


Fig. 11: Left: $\langle\sigma_z\rangle$ full dynamics (black) and approximation (orange) in first revival region; Right: Revival times over coupling for different α, Lines: Approximation, Circles: Full dynamics

3.2 Small oscillator frequency

- Influence of ground state phase transition is observable in spin dynamics with a 'relaxed' initial oscillator state

Static oscillator ($\Omega = 0$)

- Spin rotating in magnetic field $\vec{B} = \Delta\vec{e_x} + 2\gamma^2\vec{e_z}$
  - spin "localizes" similar to ground state phase transition
  - $\gamma > \gamma_c$: $\langle\sigma_z\rangle_t > 0$ for all times

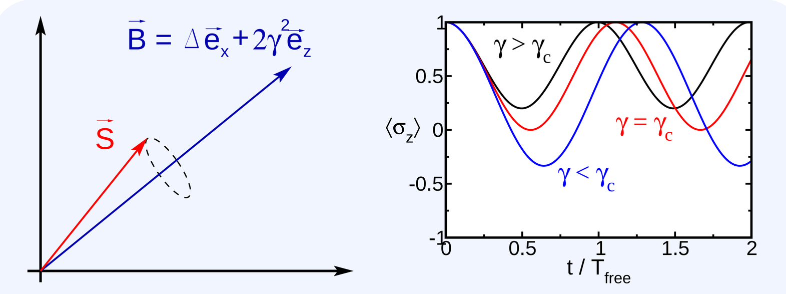


Fig. 12: Left: Rotating spin; Right: $\langle\sigma_z\rangle_t$ for different coupling

Slow oscillator $\Omega \ll \Delta$

- Crossover in spin behaviour
  - $\gamma < \gamma_c$: spin and oscillator propagates on different time scales
  - $\gamma > \gamma_c$: "lock-in" of spin and oscillator on short time scale, between: complicated oscillations

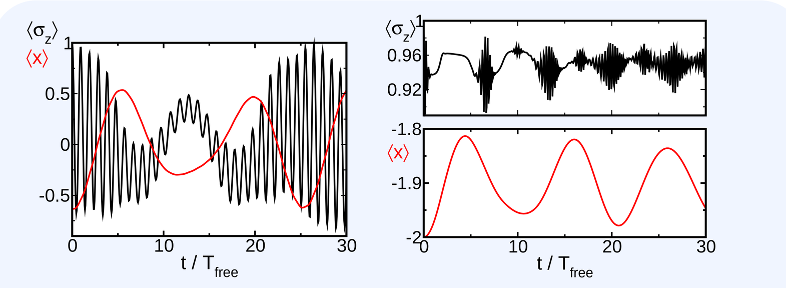


Fig. 13: $\langle\sigma_z\rangle_t$ (black) and $\langle x\rangle_t$ (red) propagating from the initial state $|\Psi\rangle_0 = |\uparrow, -\frac{\gamma}{\sqrt{\Omega}}\rangle$ with $\Omega = 0.1$. Left: $\gamma^2 = 0.1$, right: $\gamma^2 = 1$

Contrast: $\Omega = 0 \longleftrightarrow \Omega \neq 0$

- Observing the limit $\Omega \to 0$ in the dynamics
$\Omega \neq 0$: $\langle\sigma_z\rangle_t = 0$ for some time $t = T_0$
  - $\gamma < \gamma_c$: T_0 reaches the finite $\Omega = 0$ value continuously as $\Omega \to 0$
  - $\gamma > \gamma_c$: while $T_0 = \infty$ for $\Omega = 0$, $T_0 < \infty$ for $\Omega \neq 0$, but T_0 diverges as $\Omega \to 0$

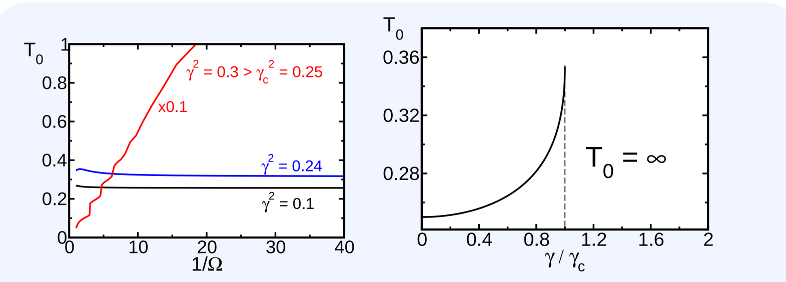


Fig. 14: Left: "Crossing time" T_0 in the limit $\Omega \to 0$ for different coupling; Right: T_0 for $\Omega = 0$

4 Conclusion

Despite its seeming simplicity, the Rabi model is a prototypical example for fundamental physical effects occuring in quantum-classical phase transitions and in the complex quantum dynamics of competing timescales. The present work thus provides a starting point for further investigations of these and related phenomena in general spin-boson models.

- Rich physics & Fundamental concepts:
  - classical phase transition vs. quantum precursor
  - renormalization of effective and real subsystem dynamics
  - influence of ground state phase transition on dynamical behaviour
  - collapses and revivals: Different effects appear on different time scales
- Relevance for modern applications
  - qBit manipulation
  - realization of strong coupling regime (e.g. Josephson junctions)
  - cQED beyond RWA