Nonlinear dynamics and quantum multistability of optomechanical systems

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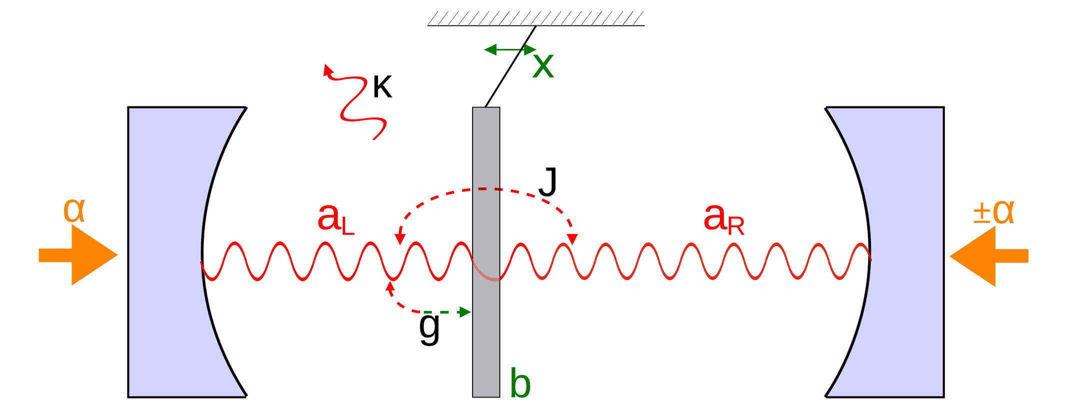
Abstract

We study the dynamics of two optomechanical systems, the cavity-cantilever and the membrane-in-the-middle setup, with a particular focus on the nonlinear classical dynamics and the transition into the quantum regime. We start with the analysis of classical dynamics, where we identify the route to chaos and the multistability of solutions, which manifests itself through the coexistence of several stable orbits at different amplitudes. Then, we study with the method of quantum state diffusion, how this optomechanical multistability is realized in the quantum regime. There, new dynamical patterns appear because quantum trajectories are affected by quantum noise and can move between different classical orbits. We explain the resulting quantum dynamics from the phase space point of view, and provide a quantitative description in terms of autocorrelation functions. In this way we can identify clear dynamical signatures of the crossover from classical to quantum mechanics in experimentally accessible quantities. Finally, we discuss a possible interpretation of our results in the sense that quantum mechanics protects optomechanical systems against the chaotic dynamics realized in the classical limit.

Model



Setup. The *membrane-in-the-middle setup* is a vibrating cantilever b with permeability J and natural frequency Ω subjected to the cavity photon field a_L and a_R with frequency Ω_{cav} via radiation pressure g. The system is driven from both sides with a laser α of frequency Ω_{las} (phase shift $\sigma = \pm 1$), while the dissipation of the photon field is taken into account with κ. The dissipation of the cantilever is neglected. The *membrane-in-the-middle setup* (MIM) represents an extension of the conventional *cavity-cantilever setup* (CC), which is irradiated only from the left with $J = a_R = 0$. For the classical dynamics of MIM we set $\kappa = \Omega = \alpha = -\sigma = 1$. For the discussion of quantum dynamics of CC we rescale the classical equations of motion with $a \to a/2\alpha$, $b \to b \cdot g$ in order to vary the system size, such that $\kappa = \Omega = 1$ and $\alpha = P\Omega^4/8g^2$ with $P = 1.5$.

Hamiltonian $\quad H = H_0 + H_{\rm int} + H_{\rm drive}$ $\quad$ *membrane-in-the-middle*

$$H_0 = \Delta\left(a_L^\dagger a_L + a_R^\dagger a_R\right) + \Omega b^\dagger b, \qquad \Delta = \Omega_{\rm las} - \Omega_{\rm cav}$$

$$H_{\rm int} = J\left(a_L^\dagger a_R + a_L a_R^\dagger\right) + g\left(b^\dagger + b\right)\left(a_L^\dagger a_L - a_R^\dagger a_R\right)$$

$$H_{\rm drive} = \alpha\left(a_L + a_L^\dagger\right) + \sigma\alpha\left(a_R + a_R^\dagger\right)$$

Quantum optical master equation at $T = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}\left[H, \rho\right] + 2\kappa \sum_{L/R}\left[a_{L/R}\rho a_{L/R}^\dagger - \frac{1}{2} a_{L/R} a_{L/R}^\dagger \rho - \frac{1}{2}\rho a_{L/R} a_{L/R}^\dagger\right]$$

Classical dynamics

$$\dot{x} = \Omega p \qquad \dot{a}_L = \left[i\Delta - igx - \kappa\right] a_L - iJa_R - i\alpha$$

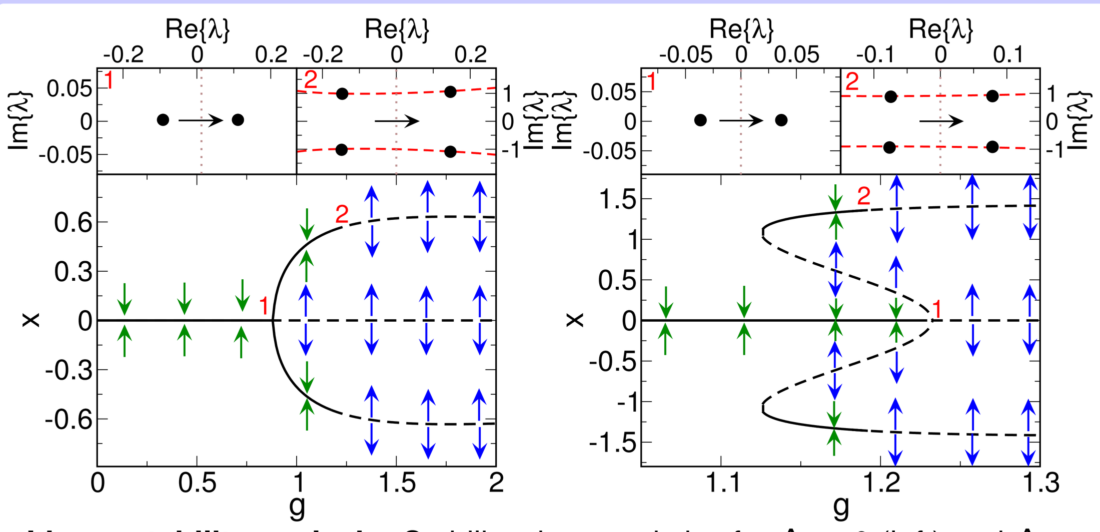
$$\dot{p} = -\Omega x - g\left(|a_L|^2 - |a_R|^2\right) \qquad \dot{a}_R = \left[i\Delta + igx - \kappa\right] a_R - iJa_L - \sigma i\alpha$$

Quantum dynamics (Quantum State Diffusion)

$$|\mathrm{d}\psi_m\rangle = -\mathrm{i}H\,|\psi_m\rangle\,\mathrm{d}t + 2\kappa \sum_{L/R}\left[\langle a_{L/R}^\dagger\rangle a_{L/R} - \tfrac{1}{2} a_{L/R}^\dagger a_{L/R} - \tfrac{1}{2}\langle a_{L/R}^\dagger\rangle\langle a_{L/R}\rangle\right]|\psi_m\rangle\,\mathrm{d}t + 2\kappa\sum_{L/R}\left[a_{L/R} - \langle a_{L/R}\rangle\right]|\psi_m\rangle\,\mathrm{d}\xi_m$$

$\rho = \text{mean}\,|\psi_m\rangle\langle\psi_m|$ $\quad$ quantum trajectory $\quad$ stochastic increment

Nonlinear Dynamics

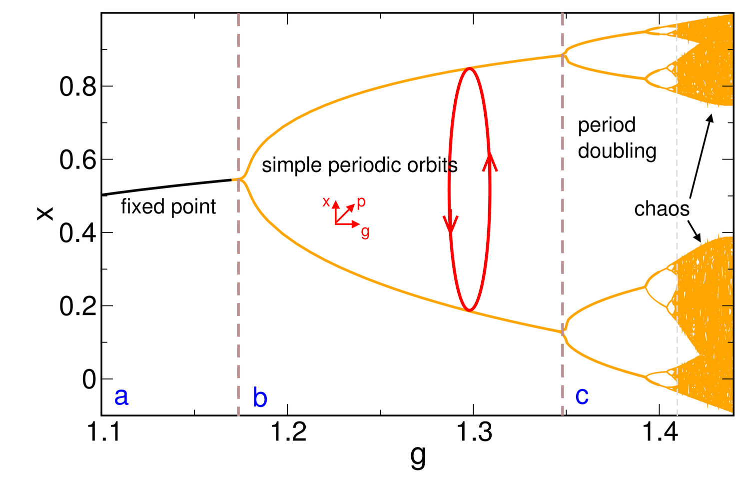


Linear stability analysis. Stability characteristics for $\Delta = 0$ (left) and $\Delta = -1.65$ (right) with $J = 0.5$. Stable (unstable) fixed points are marked with green (blue) arrows. The red cases correspond to supercritical (left) and subcritical (right) pitchfork bifurcations (1) and Hopf bifurcations (2), which are characterized by the number of zero crossings of their real parts of the eigenvalues from linear stability analysis (see the insets).

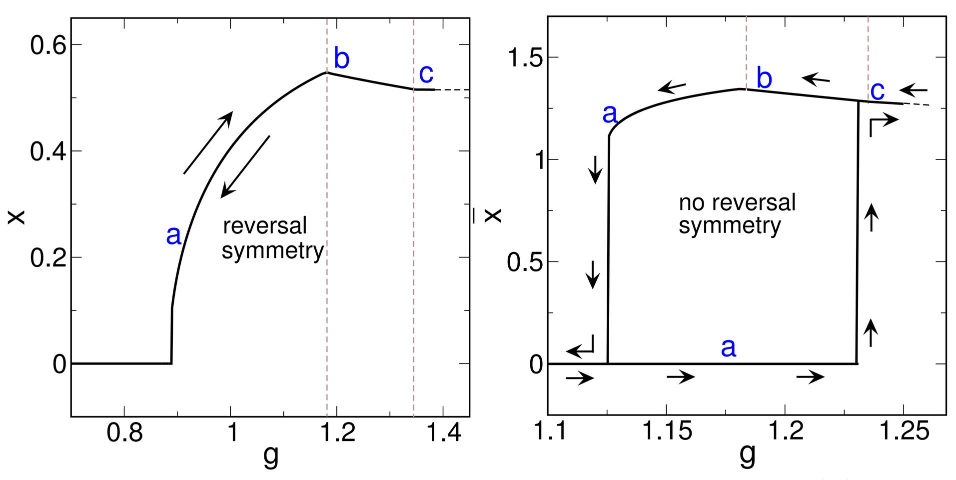
Linear stability analysis
- 5 fixed points
  $x_0 = 0$, $x_1 = -x_2$, $x_3 = -x_4$
- pitchfork bifurcations 1 and Hopf bifurcations 2

Route to chaos
(a) fixed points
↓
(b) simple periodic orbits
↓
(c) period doubling
↓
chaos

Hysteresis
- state of system $\lim_{t\to\infty}\langle x\rangle_t = \bar{x}$ depends on its history



Route to chaos. Feigenbaum diagram starting with the upper fixed point after supercritical pitchfork bifurcation with fixed points (a), simple oscillations (b) and period doublings (c), resulting finally in chaos.



Hysteresis. Dynamical evolution represented by $\lim_{t\to\infty}\langle x\rangle_t = \bar{x}$ for $\Delta = 0$ (left, supercritical) and $\Delta = -1.65$ (right, subcritical) in the control parameter space of g. Depending on the stability characteristics there is a reversal symmetry or not.

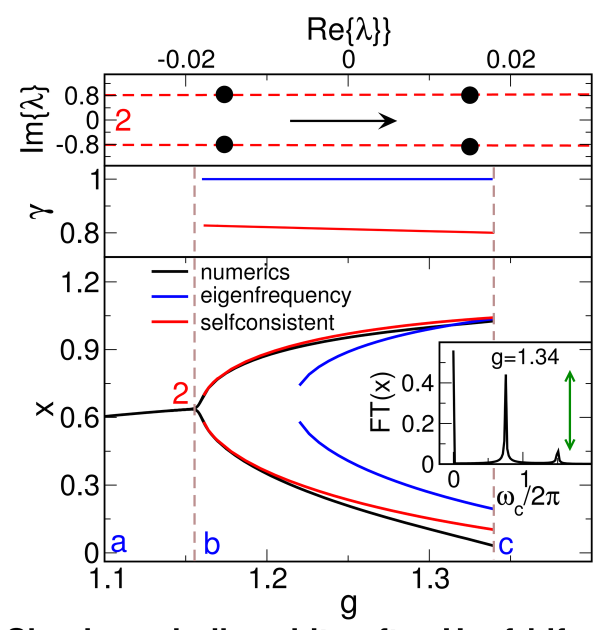
Selfsustained Oscillations

Selfconsistent ansatz

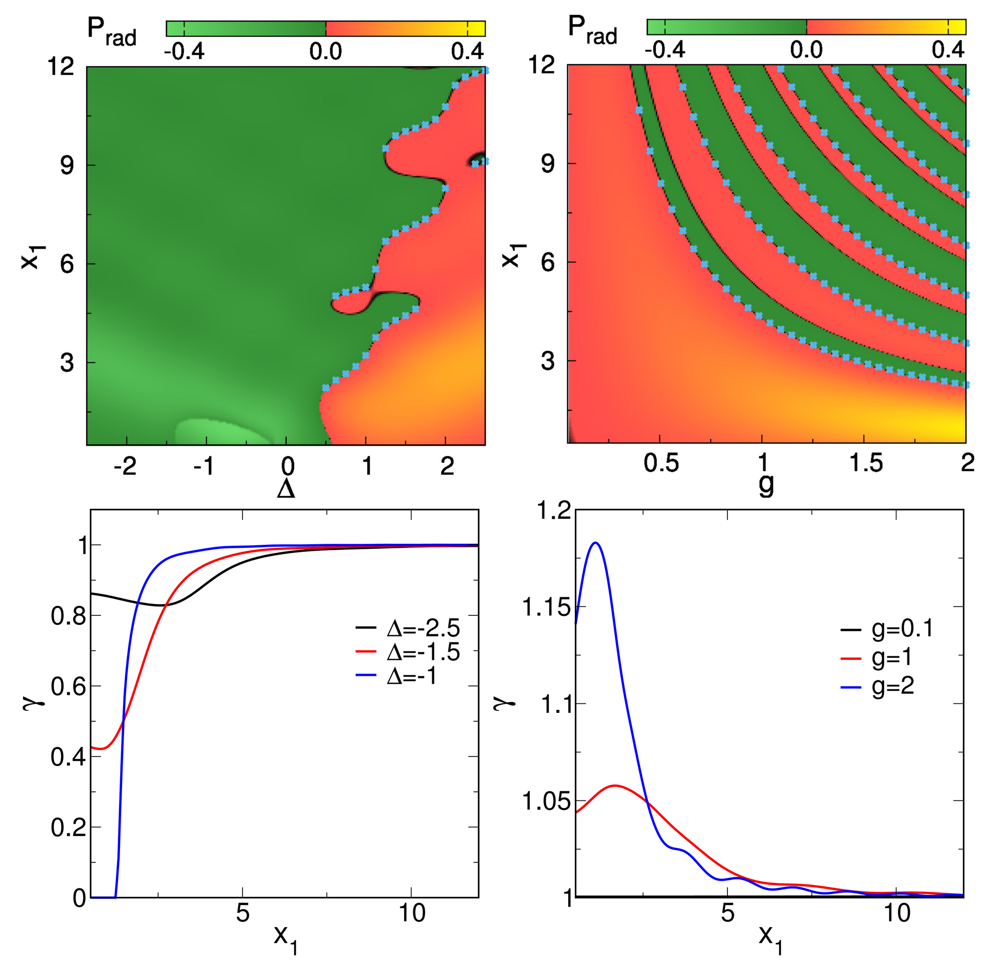
$$x(t) = x_0 + x_1 \cos\gamma\Omega t$$

Simple periodic orbits after Hopf bifurcation

Multistability of simple periodic orbits



Simple periodic orbits after Hopf bifurcation obtained from numerics (black), from the simpler eigenfrequency ansatz ($\gamma = 1$, blue) and from the full selfconsistent ansatz ($\gamma \neq 1$, red) for $J = 0, \Delta = -0.5$. Since the self-induced oscillation is not initiated with the natural frequency of the cantilever, the eigenfrequency approximation fails, whereas numerics and selfconsistent approach are in good agreement; the deviation from numerics at $g = 1.34$ close to the period doubling is due to the presence of another, not negligible, Fourier mode (see the inset).



Multistability of simple periodic orbits. Radiation power $P_{\rm rad}$ from the selfconsistent ansatz with $x_0 = 0$, for $J = 0.5$ and $g = 1$ (left) or $\Delta = 1.4$ (right), respectively. Simple periodic orbits are possible for $P_{\rm rad} = 0$ and $\mathrm{d}P_{\rm rad}/\mathrm{d}x_1 < 0$ ('power balance') with excellent agreement by comparison with numerics (blue points). Below: frequency γ from the selfconsistent ansatz for $g = 1$ (left) and $\Delta = 1.4$ (right).

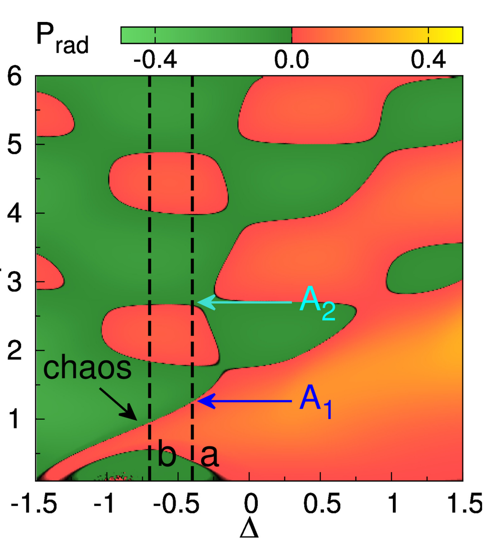
Quantum Dynamics & Quantum Multistability

Quantum-Classical Scaling Parameter

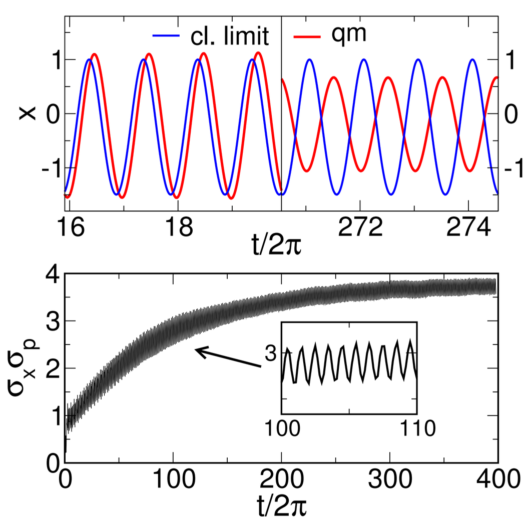
$$\sigma = g/\kappa$$

$\sigma = 0$ $\quad$ **system size** $\sim 1/\sigma$ $\quad$ $\sigma > 0$

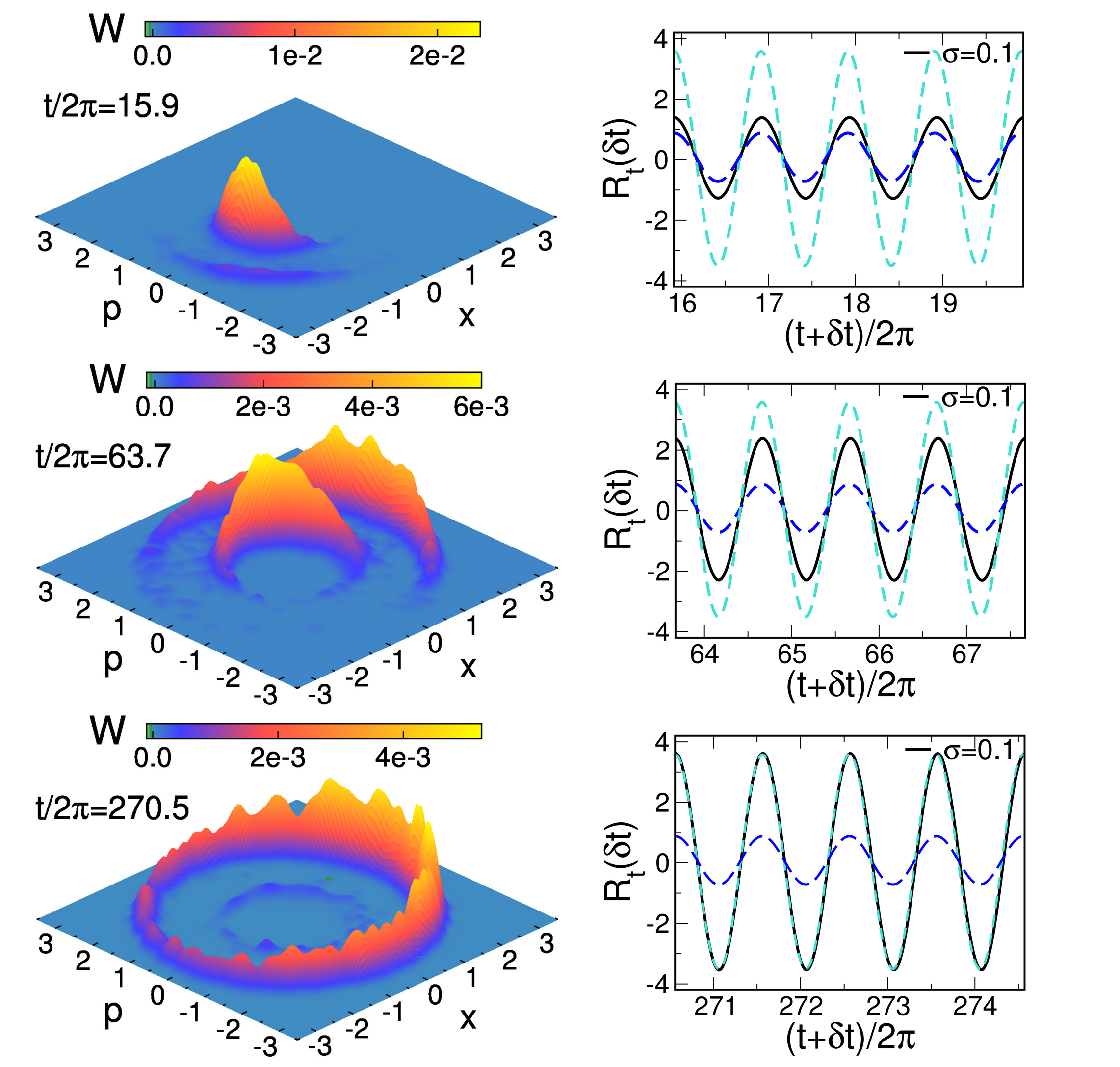
Classical Dynamics $\quad$ Quantum Dynamics



Classical Dynamics is dominated by self-induced oscillations, which lead to simple periodic orbits with different amplitudes A_1, A_2, etc. These can be located by using $P_{\rm rad} = 0$ & $\mathrm{d}P_{\rm rad}/\mathrm{d}x_1 < 0$. Marked are the two cases (a) ($\Delta = -0.4$) and (b) ($\Delta = -0.7$). For (b) there exist an area, where the selfconsistent ansatz is insufficient due to chaotic dynamics.



The **Quantum Dynamics** (for case (a) at $\sigma = 0.1$) closely follows the classical oscillations for an initial period of time, before it deviates significantly at later times. Deviations occur because the quantum state spreads out in phase space, as witnessed by the growth of the uncertainty product, whereby the cantilever position is smeared out.

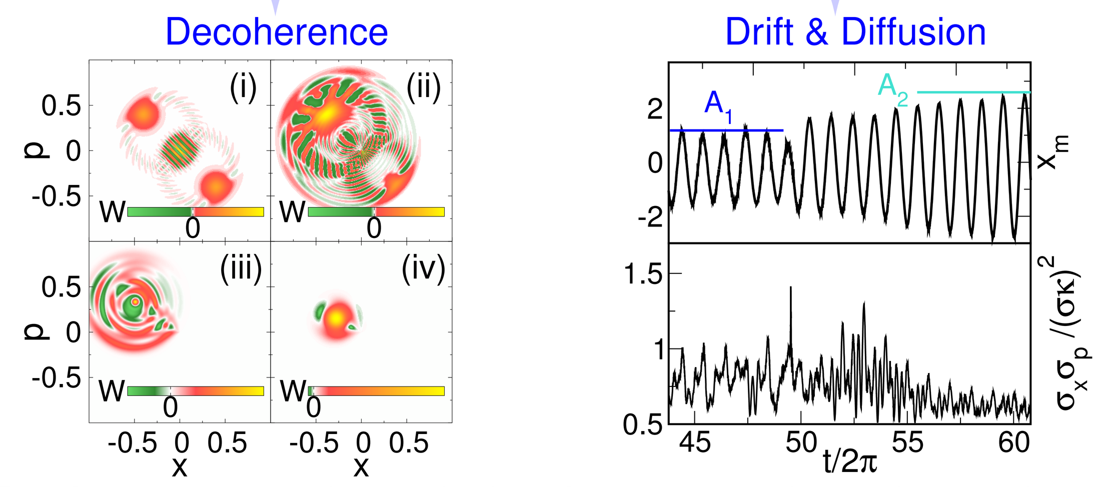


Wigner functions $W(x,p)$ in cantilever phase space and cantilever position autocorrelation functions $R_t(\delta t)$ for case (a) at $\sigma = 0.1$ slightly away from the classical limit at different points in time (dashed curves: autocorrelation functions for the two inner classical orbits). **Quantum Multistability** is the weighted localization of quasi probability density on classical orbits. A quantitative description is given by the autocorrelation function, which represents the weighted sum of the oscillatory motion on the two inner orbits (the effect is not visible by using simple expectation values, since the state is spreaded out in phase space).

Quantum State Diffusion

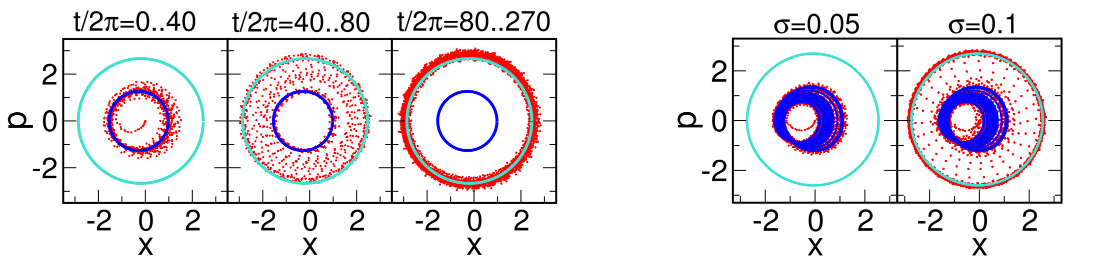
"Quantum Noise"

Decoherence $\quad$ Drift & Diffusion



Decoherence causes the localization of a quantum trajectory into a coherent state (see Wigner functions starting from a "Schrödinger cat" state), whereas **Drift & Diffusion** causes an out-spreading in phase space. Their interplay leads to "Quantum Noise", which results in fluctuations of x_m and $\sigma_x\sigma_p$ (case a, $\sigma = 0.1$).

Multistability of Quantum Trajectories $\quad$ Protection against Chaos



Stroboscopic phase space plot of a Quantum Trajectory (red dots). Quantum Noise leads to the **Multistability of Quantum Trajectories** (left, case a), which is also causing the **Protection against Chaos** (right, case b at $t/2\pi = 158$). Quantum Multistability is an effect of time scale, which is increasing if quantum noise $\sim\sigma$ is decreasing, and therefore finally vanish in the classical limit $\sigma = 0$.

[1] C. Wurl, A. Alvermann, and H. Fehske, unpublished

[2] C. Schulz, A. Alvermann, L. Bakemeier, and H. Fehske, EPL **113**, 64002 (2016)

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